Efficient investment, search, and sorting in matching markets*

Alp Atakan,† Michael Richter,‡ and Matan Tsur§

December, 2025

Abstract

We study markets where heterogeneous agents first make investment decisions and then engage in costly search to form productive matches. The trading process is random search and bargaining with explicit search costs. Despite potential hold-up and matching problems, we prove that the constrained efficient allocation is an equilibrium: the agents' private incentives to invest and to accept/reject potential partners as they search are perfectly aligned with the social benefit. Furthermore, we establish a new sorting result for two-sided markets, equilibrium existence, and conditions for uniqueness.

1 Introduction

This paper studies markets in which participants first make investment decisions and then engage in costly search to form productive matches. These are two central features in various settings. For example, in the marriage market, individuals make premarital investments in their education and career before looking for a partner. In the labor market, workers acquire human capital before searching for jobs, while firms adopt technologies before hiring workers. Likewise, in the real estate market, developers often build before finding prospective buyers; in venture capital markets, entrepreneurs invest time and money developing start-ups prior to seeking funding; and in product markets, buyers and sellers make ex-ante investments before meeting.

In such settings, agents are usually heterogeneous, and the output each pair produces depends on their prior investments. Therefore, the investment and search decisions should be studied together in equilibrium as they are mutually dependent. For example, when workers acquire skills for the job market, they consider the technologies that firms adopt and how long it will take to find jobs. In turn, the firms' investment and hiring decisions depend on the skills in the hiring pool, which itself depends on the

*We thank Jan Eeckhout, Juha Tolvanen, Daniel Garcia, Maarten Janssen, Karl Schlag, Philipp Schmidt-Dengler, and Dhruva Bhaskar for helpful comments and discussions. We acknowledge financial support from the Leverhulme Trust, CUNY-PSC, and the Austrian Science Fund: P 30922.

†Department of Economics, Queen Mary, University of London.

‡Departments of Economics, Baruch College and Royal Holloway, University of London.

§Department of Economics, University of Bristol.

workers' investments. We aim to address fundamental questions for such settings: How do agents invest? Who matches with whom and how long do they search? When is the market efficient?

There are two main obstacles to efficiency. Since agents invest before meeting their partners, a potential hold-up problem may reduce the incentive to invest. In addition, since the investment decisions vary across agents in the population, there is a potential matching problem: some agents may search too little and accept inefficient matches or search too much and reject efficient matches or both. Building on the foundational Diamond-Mortensen-Pissarides model , the prevailing view in the literature is that efficiency fails in markets with search frictions: there is under-investment (see, e.g., Acemoglu 1996) and mismatching (see, e.g., Shimer and Smith 2001).

In this paper, we contribute to the literature by developing a tractable model to study investment, search, and matching together. We depart from those papers by considering an explicit cost per search rather than discounting. Contrary to the common view in the literature, we prove a new efficiency result: the constrained efficient allocation is an equilibrium outcome. The agents' private incentives to invest and to accept/reject potential partners as they search are perfectly aligned with the social benefit. We then turn to analyzing the equilibrium structure: we prove a new sorting result, establish equilibrium existence and uniqueness results. The model and results show that our framework can serve as a workhorse for studying these markets.

In our model, there are two populations of agents, which we call buyers and sellers, but one can equally consider workers and firms, men and women, or any other two groups that invest and then match. What is important is that output is produced by pairs of agents, one from each side of the market. The model has two key ingredients. First, agents invest in skills before entering the market and they are heterogeneous in their investment costs. Second, buyer-seller pairs produce output according to the skills that they have acquired, but there is some sand in the wheels of the market: to form productive matches, the agents must engage in costly search.

We consider the standard random search and bargaining process with an explicit cost per search and without discounting, as in Atakan [2006]. Time is discrete and utility is transferrable. In every period, a new cohort of agents is born, acquires skills, and then enters the market. When two agents meet in the matching market, they can either accept each other and Nash bargain over the joint output or reject and continue searching for a better match. We analyze a *steady-state equilibrium* where, for every skill, the inflow of agents to the market equals the outflow.

The term ”skill” refers to investments that enhance productivity. For instance, in the labor market, a worker's skill is their education level, while a firm's skill is their technology. In a product market, a seller's investment reduces their production cost, a buyer's investment increases their utility, and the match output is the buyer's utility minus the seller's cost. In the marriage market, we assume that men and women are ex-ante identical – they can acquire the same skills and have the same cost distribution.

The market is competitive in that every skill has a value and agents optimize given these values. The key feature is that these values serve *double duty*: creating incentives to invest and to accept/reject matches as they search. First, regarding investment, each agent compares their marginal cost of acquiring a skill to its marginal value in the market. Second, regarding search, two agents will accept (or reject) each other whenever the match output is greater (resp. smaller) than the sum of their values. As is standard, in an equilibrium, these values are endogenously determined and must be self-consistent.

Despite the potential inefficiencies, we prove that every constrained efficient allocation is an equilibrium outcome.[1] The proof constructs market values that satisfy the standard equilibrium conditions while perfectly aligning the agents' incentives with the planner. These values simultaneously solve the investment and matching problems. This theorem also establishes the existence of equilibrium.

Notice that agents' decisions impose externalities on each other. Regarding investment, when the social planner increases some agents' investments, it directly affects their productivity and search costs, but there is also an *indirect effect* on other agents via the change in the steady-state skill composition. Regarding matching, when the planner decides that two skills should reject rather than accept, the planner forgoes their match output and incurs a higher search cost to form more productive partnerships, but must also consider the change in the steady-state skill composition. In contrast, in equilibrium, each agent invests and accepts or rejects partners solely by their private incentives, as determined by the value of each skill in the market. Remarkably, the equilibrium values make the agents internalize the direct and indirect effects.

Our second main result is that the equilibria have a clear and simple structure. We prove that there is assortative matching if the production function is super/submodular. Furthermore, if the production function is additively separable, then the equilibrium is unique and it achieves the first-best allocation. Economies with non-separable production functions can have multiple equilibria and the agents may fail to coordinate on the efficient one and so there is scope for policy interventions.[2]

Finally, we generalize our model by considering different search costs and bargaining weights, and any constant-returns-to-scale meeting function. Even in economies with homogeneous agents and no investments, it is well known that unless the bargaining weight satisfies Hosios' [1990] condition, the market is imbalanced with too many buyers relative to sellers or vice-versa. The same applies to our model. However, we prove this is the only inefficiency in the market: given the market's balance ratio, there exists an equilibrium where the investments and matching decisions are socially efficient. Thus, our main result establishing efficient investments and matching are independent of the bargaining weight and search cost parameters, and the Hosios condition.

[1]The constrained efficient allocation solves the problem faced by a social planner who controls the agents' decisions while respecting the steady-state condition. Since utility is transferable, the Pareto-optimal outcomes are the constrained efficient ones.

[2]For example, a no-investment equilibrium may occur if not investing is self-reinforcing: agents do not invest because all others do not.

Related Literature

Our paper is the first to provide a general and tractable model incorporating three components: (i) random search and bargaining, (ii) matching between heterogeneous agents, and (iii) pre-entry investments. These three components have not been studied together. Table 1 summarizes the models and results of the central papers in the strands of the literature most closely related to our work: models with transferable utility and either random search or frictionless matching.

Group	Papers	Search	Matching	Investment	Results
1	Cole et al. [2001] Noldeke and Samuelson [2015]	No	Yes	Yes	Efficiency
2	Shimer and Smith [2000] Atakan [2006]	Yes	Yes	No	Sorting (single population)
	Shimer and Smith [2001]	Yes	Yes	No	Inefficiency
3	Acemoglu [1996] Masters [1998] Acemoglu and Shimer [1999]	Yes	No	Yes	Inefficiency
4	Hosios [1990]	Yes	No	No	Efficiency (for a specific bargaining weight)
5	Gale [1987] Mortensen and Wright [2002] Lauermann [2013]	Yes	No	No	Convergence to First Best
6	This paper	Yes	Yes	Yes	Constrained Efficiency + Sorting + Robustness

Table 1: Literature Comparison

The papers in group 1 extend the classical assignment model of Shapley and Shubik [1971] to settings with ex-ante investments. These models have perfect *frictionless matching* and typically find that the first-best allocation is a competitive equilibrium outcome, but there may exist additional inefficient equilibria (see also Mailath et al. 2013, Dizdar 2018, Chade and Lindenlaub 2022).[3] We show that the *constrained efficient* allocation is an equilibrium outcome in a model with search frictions.

The papers in group 2 study the random search and bargaining model with heterogeneous agents but without investment (see also Burdett and Coles 1999). As in Atakan [2006], we consider an explicit cost per search and no discounting. Our efficiency, sorting, and existence results contribute to this literature. In particular, our sorting result is for matching markets with two different populations, such as labor and product markets, whereas the sorting results in Atakan [2006] and Shimer and Smith [2000] are not.[4] In

[3]In Chade and Lindenlaub [2022], utility is not perfectly transferable and therefore the first-best is generally unattainable but there does exist a Pareto efficient equilibrium. In Elliott and Talamàs [2023], investments are typically inefficient because payoffs cannot depend upon investment costs, and thus can't simultaneously provide incentives for both sides.

[4]Atakan [2006]'s proof relies on three assumptions: i) a symmetric production function, ii) the two sides of the market are identical, and iii) symmetric equilibrium; and therefore does not extend to our environment which does not impose any symmetry assumptions on the populations, production function,

addition, establishing existence is difficult (see, e.g., Manea 2017 and Lauermann et al. 2020) and standard techniques don't apply to our model with an endogenous inflow.

Our efficiency result stands in contrast to previous results in the search literature. First, in the standard random search and bargaining model, Shimer and Smith [2001] show that agents mismatch: low-types reject too frequently while high-types accept too often. Second, in economies with *homogeneous* agents, the hold-up problem leads to under-investment (see group 3). The key difference between those models and ours is that we have an explicit cost per search whereas those models have time discounting. The discount factor introduces implicit search costs as the agents' payoffs are delayed. Since higher types have higher continuation values, they also have higher implicit search costs, which reduces their relative bargaining position and inefficiently distorts the incentive to invest and to accept/reject matches. Our results suggest that the hold-up and matching problems are not due to search frictions *per se*, but rather due to discounting.

Hosios [1990] considers a search model with *homogeneous* agents except that the two sides of the market may meet partners at different rates depending on the balance ratio in the market - the relative size of each side. In equilibrium, the balance ratio depends on the bargaining weights and a specific weight perfectly balances the market. This applies to our model as well. However, our paper is about a different problem: we study the investment and matching decisions in a model with *heterogeneous* agents. Our efficiency result is robust to the bargaining and search cost parameters: these parameters mechanically determine the balance ratio in the market, but the investment and matching decisions are constrained efficient for this balance ratio (see Theorem 3).

The papers in group 5 study whether the random search and bargaining model converges, as the discount factor $\delta \to 1$, to the frictionless Walrasian outcome.[5] Our paper shows that the constrained efficient allocation is achieved in a market with investment, search, and matching (and additive search costs). There is a large literature on search with non-transferable utility, and on directed search, but these models are less relevant to ours. Burdett and Coles [2001] consider a marriage market with premarital investments, but they assume a very specific form of *non-transferable utility* and *homogeneous* investment costs. They show that an equilibrium exists and that it is inefficient.[6] In the literature on directed search, sellers post prices to attract buyers, and the equilibrium can achieve an efficient allocation (see, e.g., Acemoglu and Shimer 1999, Shi 2001, Jerez 2017) and sorting (see, e.g., Shimer 2005, Eeckhout and Kircher 2010, Cai et al. 2025). However, the matching process and the price-determination mechanism are substantially different than in the random search and bargaining model.

---

or agents' behavior. We have provided a novel argument to establish sorting without any symmetry assumptions. Shimer and Smith (2000) consider a one-population model, and their proof of equilibrium existence and sorting uses the above symmetry assumptions. We are unaware of any paper that extends their proofs to a model with two different populations, and whether doing so is merely a technical exercise or not. In particular, the equilibria of the two-population model may qualitatively differ from the one-population model: even if the production function is symmetric and the two populations are ex-ante identical, they may invest and match differently, which cannot occur in the one-population model.

[5] Elliott and Nava [2019] investigate conditions for efficiency in thin markets.

[6] In the non-trivial case of high investment costs, agents overinvest to appeal to better partners, and they search too much. Antler et al. [2025] find inefficient matching in a marriage model with NTU.

2 The Model

There is a continuum population of buyers with types $\beta \sim F^b$ and sellers with types $\sigma \sim F^s$. Each buyer chooses one skill from a finite set $I \subset \mathbb{N}$ and each seller chooses one skill from a finite set $J \subset \mathbb{N}$. The cost of skill i to buyer β is $C^b(i,\beta)$ and the cost of skill j to seller σ is $C^s(j,\sigma)$. Output is produced by buyer-seller pairs according to their skills and is summarized by the matrix $G = [g_{ij}]$, where the entry $g_{ij} \geq 0$ denotes the output of a pair with skills i, j. Agents have transferable utility and incur a fixed per-period search cost $c > 0$.

The type distributions F^b and F^s are continuous and strictly increasing over their connected supports: $\mathscr{B} = supp(F^b) \subseteq \mathbb{R}$ and $\mathscr{S} = supp(F^s) \subseteq \mathbb{R}$. The match output g_{ij} is strictly increasing in skills. The cost functions are non-negative, strictly increasing in both arguments, bounded and continuous in the second argument. Furthermore, they satisfy increasing differences: the difference $C^b(i',\beta) - C^b(i,\beta)$ is strictly increasing in β whenever $i' > i$ and the difference $C^s(j',\sigma) - C^s(j,\sigma)$ is strictly increasing in σ whenever $j' > j$. That is, a higher skill enhances match output, but is more costly to acquire, and higher types have higher costs and higher marginal costs.

Definition. An *economy* is a tuple $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ consisting of prior distributions, skill sets, investment cost functions, the output function, and a search cost. The economy is *symmetric* if $F^b = F^s, I = J, C^b = C^s$, and $g_{ij} = g_{ji}, \forall i, j$.

Timing. Search and matching takes place in discrete time periods over an infinite horizon. In every period, a unit measure of buyers and a unit measure of sellers are born. Each newborn agent chooses a skill and then enters the matching market. Each agent in the market incurs the search cost c and randomly meets a partner. When two agents meet, they can either accept the match or continue searching in the hope of finding a better partner. If both agents accept the match, then they exit the market and divide their output according to Nash bargaining. If at least one rejects, then they both remain in the market. In the next period, a new cohort enters the market and the process repeats itself. We refer to the agents in the market as the stock population, the agents entering the market as the inflow population, and the agents exiting the market as the outflow population.

Steady State. The economy is in a *steady state* if in the stock population the measure of agents with each skill is constant over time. Therefore, for each skill, the inflow of agents equals the outflow. In a steady state, we denote the measures of skill i buyers and skill j sellers in the stock population by b_i and s_j. The total measures of buyers and sellers in the market are $B = \sum_{i\in I} b_i$ and $S = \sum_{j\in J} s_j$, and the *proportions* of skill i buyers and skill j sellers are $x_i = b_i/B$ and $y_j = s_j/S$ (notice that $B \geq 1$ and $S \geq 1$). The notation (x_i) and (y_j) denotes the profile of buyer and seller proportions. We let $z = \langle (x_i), (y_j), B, S\rangle$ be the *state variable* where the set of all state variables is $\mathscr{Z} = \Delta(I) \times \Delta(J) \times [1,\infty)^2$.

Meetings. An agent can meet at most one partner in each period and pairs meet at random. The total number of meetings per period is $\mu(B,S) = \min(B,S)$. Therefore, if the market is balanced, i.e. $B = S$, then every agent randomly draws a partner in each period. For now, we will assume that the market is balanced, and denote the market size by $N = B = S$ and the state by $z = \langle (x_i), (y_j), N \rangle$. If the market is unbalanced, agents on the long side of the market would need to be rationed, but this cannot occur in equilibrium (see Lemma 1). In Section 6.2, we extend the analysis to consider more general meeting functions.

Strategies. An agent's strategy specifies their choice of skill and which agents they accept. We assume Markov strategies. The *investment strategy* of buyer β is $\mathbf{I}^\beta : \mathcal{Z} \to I$ and that of seller σ is $\mathbf{I}^\sigma : \mathcal{Z} \to J$. The *acceptance strategy* of a buyer with skill i is $A_i^b : \mathcal{Z} \times J \to [0,1]$, which specifies the probability she accepts a seller with skill j upon meeting. For a seller with skill j, it is $A_j^s : \mathcal{Z} \times I \to [0,1]$. Note that the acceptance strategies do not depend on the agents' identities because the match output depends only on skills. To simplify, we will suppress the state variable in the strategies. It will be convenient to summarize the acceptance strategies by a matching matrix $M = [m_{ij}]$, where the element $m_{ij} = A_i^b(j) \cdot A_j^s(i)$ is the probability that buyer i and seller j both agree to match, conditional on meeting.

Remark 1. The search cost c captures various costs incurred explicitly from search. These include the opportunity cost of time (think of the man-hours firms spend screening and interviewing candidates; while candidates forgo some income, say from driving an Uber, as they go through ads, apply, and prepare to interview); flow payments and fees (subscriptions to online search platforms, hiring talent recruiters, or advertisement fees); cognitive effort costs (browsing and comparing products online for hours, or the negative mental health impact of unemployment); or even singles paying per date. In contrast, in a model with time discounting, agents incur an implicit search cost as their payoffs are delayed. Which costs are more salient depends upon the economic situation being modeled, but there are certainly situations where additive costs are predominant.[7]

2.1 Equilibrium

Every skill has a value in the market and agents optimize given the values and the steady state. We denote the values of a skill i buyer by v_i, and of a skill j seller by w_j. The profiles of buyer and seller values are (v_i) and (w_j), respectively. As is standard in the search and matching literature, we define an equilibrium using the matching matrix and values, rather than the strategies.

Definition. A *steady state equilibrium* $\langle z, M, (v_i), (w_j) \rangle$ consists of a state variable, a matching matrix, and market values satisfying conditions (1), (3), and (4) below.

The first condition is that acceptance decisions are individually optimal. When two agents with skills i and j meet, the *surplus* is $s_{ij} = g_{ij} - v_i - w_j$, and the acceptance decisions satisfies *the Individually Rational Matching* condition *(IR Matching)*:

[7]For example, when search transpires over a short period of time and does not affect the consumption date (think of the time spent searching online for a product that will be delivered tomorrow or college students applying for jobs which they will take after graduation).

$$m_{ij} = \begin{cases} 1 & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} < 0 \end{cases} \tag{1}$$

The condition is intuitive because an agent will accept a match precisely when her payoff from doing so is greater than her continuation value. When the surplus is negative, i.e. $v_i + w_j > g_{ij}$, the match is always rejected because both agents cannot receive at least their value, while when the surplus is positive, the agents will reach a mutually beneficial agreement. If the surplus is exactly zero, then m_{ij} is unrestricted, i.e. $0 \le m_{ij} \le 1$.

When two agents accept each other, each receives their own value and half of the match surplus. This division rule is the Nash bargaining solution and also is a subgame perfect equilibrium of a strategic bargaining game (see, e.g., Atakan 2006). The second condition is that the values are self-consistent, and therefore satisfy the following recursive equation:

$$\begin{aligned} v_i &= \sum_{j \in J} y_j \left[m_{ij} \left(v_i + \frac{s_{ij}}{2} \right) + (1 - m_{ij}) v_i \right] - c, \forall i \\ w_j &= \sum_{i \in I} x_i \left[m_{ij} \left(w_j + \frac{s_{ij}}{2} \right) + (1 - m_{ij}) w_j \right] - c, \forall j \end{aligned} \tag{2}$$

That is, in every period, buyer i pays the search cost c and meets seller j with probability y_j. If a match is accepted, the buyer receives her continuation value and half of the surplus, whereas if the match is rejected, she attains her continuation value v_i. Simplifying, we obtain the *Constant Surplus* equations:

$$\begin{aligned} \sum_{j \in J} y_j m_{ij} s_{ij} &= 2c, \forall i \\ \sum_{i \in I} x_i m_{ij} s_{ij} &= 2c, \forall j \end{aligned} \tag{3}$$

The investment decisions are individually optimal: $\mathbf{I}^\beta \in \arg\max_{i \in I} \; v_i - C^b(i, \beta), \forall \beta$ and $\mathbf{I}^\sigma \in \arg\max_{j \in J} \; w_j - C^s(j, \sigma), \forall \sigma$. Since the cost function satisfies strictly increasing differences, the set of cost types who choose each skill is an interval (and hence measurable). Furthermore, at most one type can be indifferent between any two skills,[8] and thus the values (v_i) and (w_j) uniquely determine the inflows (up to measure zero). Formally, we denote by $F^b(A) = \int_A dF^b$ the measure of set A according to F^b. The measure of buyers who choose skill i is $F^b\left(\{\beta : \mathbf{I}^\beta = i\}\right) = F^b\left(\{\beta : i \in \arg\max_{i' \in I} v_{i'} - C^b(i', \beta)\}\right)$, and analogously for sellers.

The final set of conditions is that the economy is in a steady state. We refer to Equations (4) as the *Inflow=Outflow* equations:

[8] If buyer $\hat{\beta}$ is indifferent between acquiring skills i and i', where $i' > i$, then all buyers $\beta < \hat{\beta}$ strictly prefer skill i' to skill i and all buyers $\beta > \hat{\beta}$ strictly prefer skill i to skill i'.

$$\overbrace{F^b\left(\left\{\beta : i \in \arg\max_{i' \in I} v_{i'} - C^b(i',\beta)\right\}\right)}^{inflow} = \overbrace{Nx_i \sum_{j\in J} y_j m_{ij}}^{outflow}, \forall i \in I \tag{4}$$

$$F^s\left(\left\{\sigma : j \in \arg\max_{j' \in j} w_{j'} - C^s(j',\sigma)\right\}\right) = Ny_j \sum_{i\in I} x_i m_{ij}, \forall j \in J$$

The inflow is the measure of buyers who choose skill i. The outflow is the measure of skill i buyers in the market, Nx_i, times the probability of exiting (each buyer meets a skill j with probability, y_j, and they accept each other with probability, m_{ij}). The seller Inflow=Outflow equations are analogous.

2.2 Equilibrium Properties

The next two lemmas will be useful. The first states that unbalanced states do not occur in equilibria.

Lemma 1. *(No Rationing) In any equilibrium, $B = S$.*

Proof. WLOG, suppose that $B \geq S$. Then, a buyer meets a seller with probability $\rho = S/B$, and a seller meets a buyer with probability 1. Therefore, the values satisfy:

$$\forall i : v_i = \rho \sum_{j\in J} y_j \left[m_{ij}\left(v_i + \frac{s_{ij}}{2}\right) + (1-m_{ij})v_i\right] + (1-\rho)v_i - c \Rightarrow \sum_{j\in J} y_j m_{ij} s_{ij} = \frac{2c}{\rho}$$

$$\forall j : w_j = \sum_{i\in I} x_i \left[m_{ij}\left(w_j + \frac{s_{ij}}{2}\right) + (1-m_{ij})w_j\right] - c \Rightarrow \sum_{i\in I} x_i m_{ij} s_{ij} = 2c$$

Therefore, since $\sum_{i\in I} x_i = \sum_{j\in J} y_j = 1$:

$$\frac{2c}{\rho} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \left(\sum_{i\in I} x_i m_{ij} s_{ij}\right) = 2c \Rightarrow B = S$$ □

The next lemma states that, in equilibrium, the agents' values are increasing and the marginal values are bounded by the expected marginal productivity.

Lemma 2. *In any equilibrium,*

$$\frac{\sum_{j\in J} y_j m_{i'j}(g_{i'j} - g_{ij})}{\sum_{j\in J} y_j m_{i'j}} \geq v_{i'} - v_i \geq \frac{\sum_{j\in J} y_j m_{ij}(g_{i'j} - g_{ij})}{\sum_{j\in J} y_j m_{ij}} > 0,\ \forall i' > i$$

$$\frac{\sum_{i\in I} x_i m_{ij'}(g_{ij'} - g_{ij})}{\sum_{i\in I} x_i m_{ij'}} \geq w_{j'} - w_j \geq \frac{\sum_{i\in I} x_i m_{ij}(g_{ij'} - g_{ij})}{\sum_{i\in I} x_i m_{ij}} > 0,\ \forall j' > j$$

In particular, if $m_{ij} = 1$, $\forall i, j$, then the marginal value equals the expected marginal productivity: $v_{i'} - v_i = \sum_{j\in J} y_j(g_{i'j} - g_{ij})$ and $w_{j'} - w_j = \sum_{i\in I} x_i(g_{ij'} - g_{ij})$.

Proof. The Constant Surplus and IR Matching conditions imply that:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = 2c = \sum_{j\in J} y_j m_{i'j} s_{i'j} \geq \sum_{j\in J} y_j m_{ij} s_{i'j}$$

Subtracting the RHS from the LHS, and normalizing:

$$v_{i'} - v_i \geq \frac{\sum_j y_j m_{ij}(g_{i'j} - g_{ij})}{\sum_j y_j m_{ij}} > 0$$

The upper bound is derived analogously by switching i and i'. □

These two Lemmas follow from the Constant Surplus Equations. Lemma 2 also implies that there is a uniform bound on marginal values: $\max_j g_{i'j} - g_{ij} \geq v_{i'} - v_i \geq \min_j g_{i'j} - g_{ij}$.

Remark 2. The Constant Surplus equations have two further implications: First, they determine the values for unchosen (measure 0) skills, and therefore we are not free to set those values arbitrarily (for instance, to minus infinity). Second, every agent has at least one partner with whom the surplus is positive. Furthermore, that partner is not of measure 0, which implies that there are no pathological equilibria where an agent searches forever.

Remark 3. If $\langle z, M, (v_i), (w_j)\rangle$ is an equilibrium, then so is $\langle z, M, (v_i + t), (w_j - t)\rangle$ for any transfer $t \in \mathbb{R}$. Therefore, there is at least one degree of freedom in the equilibrium values. We now show that there is in fact exactly one degree of freedom. This is because the marginal values, i.e. Δv_i, are uniquely pinned down by the investment decisions and a Constant Surplus equation imposes an additional condition on the value functions.

3 Illustrative Example

Consider a symmetric economy with two skills, $I = J = \{0, 1\}$. Each agent can either invest and become skilled, $i = j = 1$, or not invest and remain unskilled, $i = j = 0$. The cost of becoming skilled is the agent's type, which is uniformly distributed $\beta, \sigma \sim U[0,3]$. Denote by $x = x_1$ and $y = y_1$ the proportion of skilled buyers and skilled sellers. Production is supermodular:

$$G^{sup} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

That is, skilled-skilled pairs produce $g_{11} = 4$, unskilled-unskilled pairs produce $g_{00} = 1$, and skilled-unskilled pairs produce $g_{10} = g_{01} = 2$. Notice that the marginal productivity of becoming skilled is greater when matched with a skilled agent than when matched with an unskilled agent, $g_{11} - g_{01} = 2 > 1 = g_{10} - g_{00}$.

Equilibrium

In any equilibrium, buyers with costs $\beta \le \Delta v := v_1 - v_0$ invest and become skilled, and sellers with costs $\sigma \le \Delta w := w_1 - w_0$ invest and become skilled. By Lemma 2, the marginal values satisfy $\Delta v, \Delta w \in [1,2]$. Therefore, buyers/sellers with costs β, σ below 1 always invest and those with costs above 2 never invest. The question is: What happens for agents whose costs are between 1 and 2 and who matches with whom? There are two natural candidates for equilibria: *All Skills Match*, where all agents accept each other, and *Positive Assortative Matching* (PAM), where each agent matches only with the same skill.

Claim 1. (i) There exists an equilibrium where All Skills Match iff $c \ge 1/8$. (ii) There exists an equilibrium with PAM iff $c \le 1/8$. In both cases: $x = y = \frac{1}{2}$ and the threshold investment types are $\beta_1 = \sigma_1 = \frac{3}{2}$.

Proof. The equilibrium conditions are specified in the following table:

	All Skills Match	**PAM**
IR Matching	$m_{ij} = 1 \Rightarrow s_{ij} \ge 0$	$m_{11} = m_{00} = 1 \Rightarrow s_{11}, s_{00} \ge 0$ $m_{10} = m_{01} = 0 \Rightarrow s_{10}, s_{01} \le 0$
Constant Surplus Equations	$ys_{11} + (1-y)s_{10} = 2c$ $ys_{01} + (1-y)s_{00} = 2c$ $xs_{11} + (1-x)s_{01} = 2c$ $xs_{10} + (1-x)s_{00} = 2c$	$ys_{11} = 2c$ $(1-y)s_{00} = 2c$ $xs_{11} = 2c$ $(1-x)s_{00} = 2c$
Inflow=Outflow	$F(\Delta v) = Nx$ $1 - F(\Delta v) = N(1-x)$ $F(\Delta w) = Ny$ $1 - F(\Delta w) = N(1-y)$	$F(\Delta v) = Nxy$ $1 - F(\Delta v) = N(1-x)(1-y)$ $F(\Delta w) = Nxy$ $1 - F(\Delta w) = N(1-x)(1-y)$

Table 2: Equilibrium Conditions

Part (i): Consider an equilibrium where **All Skills Match**. By Lemma 2, the marginal values equal the marginal productivities:

$$\Delta v = y(g_{11} - g_{01}) + (1-y)(g_{10} - g_{00}) = 1 + y$$

$$\Delta w = x(g_{11} - g_{10}) + (1-x)(g_{10} - g_{00}) = 1 + x$$

Since the market clears in every period: $F(\Delta v) = x$ and $F(\Delta w) = y$. Thus, $F(1+y) = x$ and $F(1+x) = y \Rightarrow x = y = 1/2$. This condition is necessary but not sufficient for an equilibrium: we must find supporting values that solve the CS equations and satisfy the matching conditions.

Since $\Delta v = \Delta w = 1.5$ and $g_{10} = g_{01}$, we have $s_{10} = s_{01}$, and $s_{11} - s_{10} = g_{11} - g_{10} - \Delta v = 2 - \Delta v = \frac{1}{2}$, and $s_{00} - s_{10} = g_{00} - g_{10} + \Delta v = \frac{1}{2}$. Thus, skilled-unskilled pairs generate the lowest match surplus, and the CS equation $\frac{1}{2}(s_{11} + s_{10}) = \frac{1}{2}(s_{10} + s_{10} + 0.5) = 2c \Rightarrow s_{10} = 2c - \frac{1}{4}$. Thus, the matching condition $s_{10} \ge 0$ is equivalent to $c \ge 1/8$. The values can be derived from the CS equations: $v_1 = w_1 = \frac{15}{8} - c$ and $v_0 = w_0 = \frac{3}{8} - c$.

Part (ii): In any **PAM** equilibrium, the Constant Surplus Equations imply that the steady state must be symmetric $x = y$ and the Inflow=Outflow conditions imply $\Delta v = \Delta w$. Thus, subtracting the two CS equations, $s_{11} = \frac{2c}{x}$ and $s_{00} = \frac{2c}{1-x}$, and rearranging: $\Delta v + \Delta w = 2\Delta v = g_{11} - g_{00} - (\frac{2c}{x} - \frac{2c}{1-x})$ and so $\Delta v = 1.5 - \frac{c}{x} + \frac{c}{1-x}$. Adding the inflow equations yields $1 = Nx^2 + N(1-x)^2$, so $N = \frac{1}{x^2+(1-x)^2}$. This gives a fundamental equation:

$$\overbrace{F(\Delta v) = F\left(\frac{3}{2} - \frac{c}{x} + \frac{c}{1-x}\right)}^{\text{inflow}} = \overbrace{\frac{x^2}{x^2 + (1-x)^2}}^{\text{outflow}} \tag{5}$$

In this case, the fundamental equation has a unique solution $x = 1/2$ that satisfies the equilibrium conditions. Like in part (i), we need to find supporting values that solve the CS equations, and satisfy the matching conditions.

As in Part (i), since $\Delta v = \Delta w = 1.5$ and $g_{10} = g_{01}$, it holds that: $s_{10} = s_{01}$, $s_{11} = s_{00} = s_{10} + \frac{1}{2}$, and skilled-unskilled pairs generate the lowest match surplus. From the CS equations, $\frac{1}{2}s_{11} = 2c$ and so $s_{10} = 4c - \frac{1}{2}$. Clearly, $4c - \frac{1}{2} \leq 0$ is equivalent to $c \leq \frac{1}{8}$. The supporting values are $v_1 = w_1 = 2 - \frac{c}{x}$ and $v_0 = w_0 = \frac{1}{2} - \frac{c}{1-x}$. □

Remark 4. In this example, the equilibrium is unique for $c = 1/8$. However, there may be coordination issues for other cost-type distributions. If the support of F were smaller, say $F = U[1.2, 1.8]$, then in addition to the two equilibria above, there are two non-interior equilibria where: i) everyone invests and ii) no one invests. In both, all skills match. The median cost type was fixed at 3/2, which exactly equals $(g_{11} - g_{00})/2$. If this were not the case, say $F = U[0.8, 2.8]$, then the above analysis still applies but there are three qualitative differences. First, in the PAM equilibrium, the solution to the fundamental equation (5) will depend on c, and as a result, both the steady state $x(c) = y(c)$ and the investment thresholds $\beta_1(c) = \sigma_1(c)$ depend on c. The All Skills Match equilibrium is essentially the same only $x^{all} = y^{all} = 0.2$. Second, these two equilibria both exist over a non-trivial region of costs. Third, there might be one additional equilibrium, where skilled-unskilled agents match with some interior probability.

Efficiency

Comparing the equilibria above illustrates a basic tradeoff: skilled-unskilled pairs are willing to settle and accept less productive matches if the search cost is high $c > 1/8$; whereas they reject these matches and search for better ones if $c < 1/8$. Is this socially efficient in terms of the search and investment decisions?

To address this, consider a social planner who controls the agents' decisions and wants to maximize total welfare subject to steady state conditions. Formally, the planner controls the investment thresholds, matching decisions, and state to maximize per-period welfare:[9]

[9] First term: $Nx_i y_j m_{ij}$ is the measure of accepted matches between skills i, j and g_{ij} is their output.

$$\mathscr{W}(x,y,N,[m_{ij}],\beta_1,\sigma_1) = \overbrace{\sum_{i=0}^{1}\sum_{j=0}^{1} N x_i y_j m_{ij} g_{ij}}^{\text{Productivity}} - \overbrace{2Nc}^{\text{Search Cost}} - \overbrace{\int_0^{\beta_1} \beta dF(\beta) - \int_0^{\sigma_1} \sigma dF(\sigma)}^{\text{Investment Cost}}$$

subject to the steady state constraints:

$$Nx\sum_{j=0}^{1} y_j m_{1j} = F(\beta_1), \qquad N(1-x)\sum_{j=0}^{1} y_j m_{0j} = 1-F(\beta_1),$$
$$Ny\sum_{i=0}^{1} x_i m_{i1} = F(\sigma_1), \qquad N(1-y)\sum_{i=0}^{1} x_i m_{i0} = 1-F(\sigma_1)$$

Consider two simple policies.

1) **All Skills Match:** If the planner decides $m_{ij} = 1, \forall i, j$, then the steady state market size is $N = 1$, $x = F(\beta_1) = \frac{\beta_1}{3}$ and $y = F(\sigma_1) = \frac{\sigma_1}{3}$. Total welfare is

$$\begin{aligned}\mathscr{W} &= 4xy + 2y(1-x) + 2x(1-y) + (1-x)(1-y) - 2c - \int_0^{3x} \tfrac{\beta}{3} d\beta - \int_0^{3y} \tfrac{\sigma}{3} d\sigma \\ &= y + x + 1 + xy - 2c - 1.5x^2 - 1.5y^2\end{aligned}$$

The planner would always want agents with cost below 1 to invest because their marginal productivity (MP) outweighs marginal investment cost (MC). Starting at that point, the planner increases the investment threshold up to the point where $MP = MC$ which yields the optimal state, $x = y = 1/2$ and hence the optimal thresholds are $\beta_1 = \sigma_1 = 3/2$.

2)**PAM:** If the planner decides $m_{11} = m_{00} = 1$ and $m_{01} = m_{10} = 0$, then the steady state equations are:

$$Nxy = F(\beta_1) = \tfrac{\beta_1}{3}, \;\; N(1-x)(1-y) = 1-F(\beta_1)$$
$$Nyx = F(\sigma_1) = \tfrac{\sigma_1}{3}, \;\; N(1-y)(1-x) = 1-F(\sigma_1)$$

which imply $N = \frac{1}{xy+(1-x)(1-y)}$, $\beta_1 = \sigma_1 = 3Nxy$ and so the planner's optimization problem is two dimensional in (x, y). Total welfare is

$$\begin{aligned}\mathscr{W}^{PAM} &= N\left[4xy + (1-x)(1-y)\right] - 2Nc - \int_0^{\beta_1} \beta dF(\beta) - \int_0^{\sigma_1} \sigma dF(\sigma) \\ &= 1 + \frac{3xy(1-x)(1-y)}{(xy+(1-x)(1-y))^2} - \frac{2c}{xy+(1-x)(1-y)}\end{aligned}$$

If $c < 3/8$, the optimum is interior at $x = y = 1/2$ and $\beta_1 = 3/2$ and $N = 2$.

To sum up, the welfare under these two policies are

$$\mathscr{W}^{All} = 2.25 - 2c - 0.75$$
$$\mathscr{W}^{PAM} = 2.5 - 4c - 0.75$$

The Planner can choose many other policies: they can use a probabilistic matching rule or another pure matching rule, they can change the investment thresholds and the steady state. However, any other policy is dominated by one of these two. Consequently, the upper envelope of these two lines is the value of the planner's problem: the planner chooses the above All Skills Match policy if $c > 1/8$ and chooses the above PAM policy if $c < 1/8$. Figure 1 depicts the welfare of these two policies as a function of the search cost c. The trade-off is between higher productivity (PAM) and lower search costs (All Skills Match). The shaded regions are where each allocation is an equilibrium. This figure visually demonstrates a Second Welfare Theorem: the upper envelope is an equilibrium.

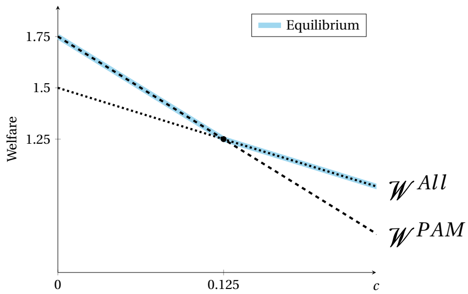

Figure 1: Equilibrium and Welfare

4 A Second Welfare Theorem for Search

To simplify notation, we label the skills as $I = \{0, 1, \ldots, |I| - 1\}$ and $J = \{0, 1, \ldots, |J| - 1\}$. The constrained efficient allocation is the solution to the problem of a social planner who chooses the investment and acceptance strategies and sets the stock in the matching market, in order to maximize per-period total welfare, subject to the condition that the economy is in a steady state. Without loss of generality: i) the planner chooses a balanced state,[10] $B = S = N$; ii) the matching strategies are represented by a matching matrix; and iii) since the investment cost functions satisfy strictly increasing differences, the planner's optimal investment strategies can be defined by thresholds $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, so that all buyers of type $\beta \in (\beta_{i+1}, \beta_i)$ choose skill i and all sellers of type $\sigma \in (\sigma_{j+1}, \sigma_j)$ choose skill j. Notice that the thresholds are descending because costs increase with type, so higher types choose lower skills. The planner chooses a tuple $\langle z, M, (\beta_i), (\sigma_j) \rangle$ of steady state, matching matrix, and investment thresholds in order to maximize:

[10] If $B > S$, then there exists another state with lower total search cost and identical output and investment cost.

$$\mathcal{W}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = \sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta)\, d\beta \qquad (6)$$

$$- \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma)\, d\sigma$$

subject to

$$flow_i^b := \left(F^b(\beta_i) - F^b(\beta_{i+1})\right) - N x_i \sum_{j\in J} y_j m_{ij} = 0, \forall i \qquad (7)$$

$$flow_j^s := \left(F^s(\sigma_j) - F^s(\sigma_{j+1})\right) - N y_j \sum_{i\in I} x_i m_{ij} = 0, \forall j \qquad (8)$$

$$x_i \geq 0, \forall i \qquad (9)$$

$$y_j \geq 0, \forall j \qquad (10)$$

$$X := 1 - \sum_{i\in I} x_i = 0 \qquad (11)$$

$$Y := 1 - \sum_{j\in J} y_j = 0 \qquad (12)$$

$$1 \geq m_{ij} \geq 0, \forall i, j \qquad (13)$$

$$F^b(\beta_{|I|}) = F^s(\sigma_{|J|}) = 0 \qquad (14)$$

$$F^b(\beta_0) = F^s(\sigma_0) = 1 \qquad (15)$$

The first term in the objective function is per-period total output (the measure of formed matches between buyer i and seller j is $N x_i y_j m_{ij}$ and the match output is g_{ij}), the second term is the per-period total search cost, and the last two terms are the per-period total investment costs. The first constraint is that inflow equals outflow. The other conditions stipulate that x_i, y_j are proportions, m_{ij} are probabilities, and that the planner must assign a skill to every agent.

Remark 5. Notice that the maximization problem does not explicitly require that $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, nor that $N > 0$, because these conditions are implied by the other constraints (see proof).

Theorem 1. ***(Second Welfare Theorem)*** *For every economy* $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$*:*

i) There exists an optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$*.*

ii) Every optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *can be decentralized. That is, there are values* (v_i^*), (w_j^*), *and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* $x_i, y_j > 0$.

The theorem demonstrates that any optimum policy can be decentralized as an equilibrium. The proof shows that the equilibrium values that decentralize the optimal allocation are the shadow values of the flow constraints in the dual problem. We show that these values are internally self-consistent with the bargaining procedure, that is, they satisfy the Constant Surplus equations; and also motivate the agents to invest and match efficiently. For instance, if the planner wants buyer β and seller σ to choose skill i^* and j^*, then $i^* \in \arg\max_{i\in I} v_i - C^b(i,\beta)$ and $j^* \in \arg\max_{j\in J} w_j - C^s(j,\sigma)$; and if the planner wants them to accept (reject) each other, then $v_{i^*} + w_{j^*} \le g_{i^*j^*}$ $(v_{i^*} + w_{j^*} \ge g_{i^*j^*})$.

Proof. First, we show that the constraints of the problem imply that $N > 0$, and $\beta_i \ge \beta_{i+1}$ for all i, and $\sigma_j \ge \sigma_{j+1}$ for all j. To see this, observe that $F^b(\beta_{|I|}) = 0$ and $F^b(\beta_0) = 1$, and so there exists a skill i such that $F(\beta_i) > F(\beta_{i+1})$. By constraint $flow_i^b$, it must be that $Nx_i \sum_{j\in J} y_j m_{ij} > 0$. Since x_i, y_j, m_{ij} are all non-negative, it follows that $N > 0$. Thus, the outflow of every skill is non-negative, and from the flow conditions, it must be that $\beta_i \ge \beta_{i+1}$ for all i, and likewise $\sigma_j \ge \sigma_{j+1}$ for all j.

(i) Existence: To demonstrate existence, since the objective is continuous, all we need to show is that the policy space is compact. First, there is a uniform upper bound $\overline{N}$ so that in any optimum, $N \le \overline{N}$ (recall that $N \ge 0$). For the upper bound, notice that the Inflow=Outflow constraints imply $\sum_{i\in I}\sum_{j\in J} Nx_i y_j m_{ij} = 1$, and therefore the first term in the welfare expression is a convex combination of g_{ij} and therefore is uniformly bounded by $\max g_{ij}$. Thus, $\lim_{N\to\infty} \mathscr{W} = -\infty$ and so the optimal policy cannot involve arbitrarily large N. The planner can choose quantiles $F(\beta_i)$ instead of thresholds β_i, and since the objective is also continuous in the quantiles and the quantile space is bounded, a maximum indeed exists.

(ii) Decentralizing optimal allocations: The dual problem is

$$\begin{aligned}\mathscr{L}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = &\sum_{i\in I}\sum_{j\in J} Nx_i y_j m_{ij} g_{ij} - 2Nc \\ &- \sum_{i\in I}\int_{B_i} C^b(i,\beta) f^b(\beta) d\beta - \sum_{j\in J}\int_{S_j} C^s(j,\sigma) f^s(\sigma) d\sigma \\ &+ \sum_{i\in I} v_i \cdot flow_i^b + \sum_{j\in J} w_j \cdot flow_j^s + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\ &+ \sum_{i\in I}\sum_{j\in J}\left(\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1 - m_{ij})\right)\end{aligned}$$

We will first show that a constraint qualification holds and then construct an equilibrium using the shadow values from the KKT conditions.

1) The Constraint Qualifications: Since the problem is not convex, we use the constant rank regularity condition, which requires that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in the vicinity of the optimal point is constant (Janin [1984]). The formal proof is in Lemma 3 in the Appendix.

2) Deriving values from the KKT conditions: Due to the constraint qualification above, the first order conditions (FOC) of the dual problem $\mathscr{L}$ are necessary at any optimum:

$$\text{FOC(N):} \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2c - \sum_{i\in I} v_i \left(x_i \sum_{j\in J} y_j m_{ij}\right) - \sum_{j\in J} w_j \left(y_j \sum_{i\in I} x_i m_{ij}\right) = 0$$

$$\iff \sum_i \sum_j x_i y_j m_{ij}(g_{ij} - v_i - w_j) = 2c$$

$$\text{FOC}(x_i)\text{: } N\sum_j y_j m_{ij} g_{ij} - v_i N \sum_j y_j m_{ij} - N\sum_j w_j m_{ij} y_j - \gamma + \phi_i = 0$$

$$\iff N\sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = \gamma - \phi_i$$

$$\text{FOC}(y_j)\text{: } N\sum_i x_i m_{ij} g_{ij} - N\sum_i v_i x_i m_{ij} - w_j N \sum_i m_{ij} x_i - \lambda + \psi_j = 0$$

$$\iff N\sum_i x_i m_{ij}\left(g_{ij} - v_i - w_j\right) = \lambda - \psi_j$$

Complementary slackness: $\phi_i x_i = 0$ and $y_j \psi_j = 0$ and $\phi_i, \psi_j \geq 0$.

$$\text{FOC}(m_{ij})\text{: } N x_i y_j g_{ij} - v_i N x_i y_j - w_j N x_i y_j + \eta_{ij} - \hat{\eta}_{ij} = 0$$

$$\iff N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

Complementary slackness: $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

$$\text{FOC}(\beta_i)\text{: } f^b(\beta_i)(v_i - v_{i-1}) = f^b(\beta_i)\left(C^b(i,\beta_i) - C^b(i-1,\beta_i)\right), \text{ for } i \in \{1,\ldots,I-1\}$$

$$\text{FOC}(\sigma_j)\text{: } f^s(\sigma_j)(w_j - w_{j-1}) = f^s(\sigma_j)\left(C^s(j,\sigma_j) - C^s(j-1,\sigma_j)\right), \text{ for } j \in \{1,\ldots,J-1\}$$

We now show that the shadow values v_i, w_j, together with the matching matrix M and state z, constitute an equilibrium.

Decentralizing the constrained optimal allocation when z is interior (ii): To verify the Constant Surplus equations, notice that:

$$N \cdot 2c = N\sum_I \sum_J x_i y_j m_{ij}(g_{ij} - v_i - w_j) = \sum_I x_i N \sum_J y_j m_{ij}(g_{ij} - v_i - w_j)$$
$$= \sum_I x_i(\gamma - \phi_i) = \sum_I \gamma x_i - \phi_i x_i = \sum_I \gamma x_i = \gamma$$

The first line uses FOC(N), while the second line uses FOC(x_i), complementary slackness ($\phi_i x_i = 0$), and the condition $\sum_I x_i = 1$. Therefore $\gamma = 2cN$. Since z is interior, $\phi_i = 0$, and the FOC(x_i) is $\sum_J y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = 2c$, which is the Constant Surplus equation for skill i. An analogous argument holds for the sellers' Constant Surplus equations.

To verify the IR Matching conditions, notice that if $g_{ij} - v_i - w_j > 0$, the FOC for m_{ij} requires that $\hat{\eta}_{ij} > 0$ and therefore $m_{ij} = 1$. Similarly, if $g_{ij} - v_i - w_j < 0$, the FOC for m_{ij} requires that $\eta_{ij} > 0$ and therefore $m_{ij} = 0$.

To verify that the investments are incentive compatible, we show that for any type $\beta \in [\beta_{i+1}, \beta_i]$, their most preferred skill is i. To see this, for any lower skill, $i' \leq i$, the FOC for the threshold $\beta_{i'}$ is $f(\beta_{i'})(v_{i'} - v_{i'-1}) = f(\beta_{i'})(C^b(i',\beta_{i'}) - C^b(i'-1,\beta_{i'}))$ and recall that $\beta_{i'} \geq \beta$. Since $f > 0$ everywhere, this can be simplified to $v_{i'} - C^b(i',\beta_{i'}) = v_{i'-1} - C^b(i'-1,\beta_{i'})$. Since type $\beta_{i'}$ is indifferent between the skills i' and $i'-1$, by single-crossing, type β weakly prefers skill i' to skill $i'-1$. Thus, type β weakly prefers i to any lower skill i'. An analogous argument applies for higher skills.

The case of a non-interior z can be found in the Appendix. □

The proof of this welfare theorem uses the oldest trick in the book: to find the market values (or prices), we look at the shadow values of the inflow=outflow constraints in the planner's problem. Surprisingly, these shadow values satisfy the Constant Surplus equations. It immediately follows from Theorem 1 that an equilibrium exists.

Corollary 1. *An equilibrium exists.*

The following proposition demonstrates some comparative statics for welfare.

Proposition 1. *The welfare function $\mathcal{W}$ is continuous, strictly decreasing, and convex in c. Moreover, the population size N is weakly decreasing in c.*

The proof is in the Appendix. It relies on the observation that $\partial \mathcal{W}/\partial c = -2N$, which follows immediately from the envelope theorem, implying that a shock to c has greater impact on welfare when c is small than when c is large.

Remark 6. (Matching and Values of Unrealized Skills) Theorem 1 proves that any optimum can be decentralized (modulo matching between unrealized skills). The planner can match unrealized types in any fashion because they have no impact on welfare, and thus the optimization problem places no restriction on their matching. However, the equilibrium conditions (the Constant Surplus equations and IR Matching conditions) apply for all skills, including unrealized ones. In the Appendix, we construct the matching and values for these unrealized skills.

4.1 Outside Options and Endogenous Entry

We now extend the efficiency result to the case where agents have outside options. Suppose that every new-born agent can either invest and enter the market or opt out and receive the outside payoff equal to u^b for buyers and u^s for sellers. In equilibrium, buyer β enters the market if and only if $\max_i v_i - C^b(i,\beta) \geq u^b$, and seller σ enters if and only if $\max w_j - C^s(j,\sigma) \geq u^s$. We focus on the interesting case where there are gains to trade, and so for at least two types, β and σ, $\max_{i\in I, j\in J} g_{ij} - 2c - C^b(i,\beta) - C^s(j,\sigma) > u^b + u^s$. The only difference from the baseline model is that the planner now also chooses the entry thresholds β_0 and σ_0 in order to maximize:

$$\mathcal{W} = N\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta)\, d\beta - \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma)\, d\sigma$$
$$+ \int_{\beta_0}^{\infty} u^b f^b(\beta)\, d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)\, d\sigma$$

and the boundary conditions $F^b(\beta_0) = 1$ and $F^s(\sigma_0) = 1$ are removed.

Corollary 2. *In a model with outside options, the constrained efficient outcome is an equilibrium.*

The proof shows that the shadow values still constitute an equilibrium (see Appendix). As before, v_0 is the shadow value of the skill 0 flow constraint. However, there is an additional first-order condition since β_0 is now endogenous: $v_0 - C^b(0, \beta_0) = u^b$ which is precisely the equilibrium entry condition for buyers. An analogous argument holds for sellers.

Remark 7. In the baseline model, there is exactly one degree of freedom in the equilibrium values (see Remark 3). In the model with outside options, there is an additional entry condition and thus the values are unique.

5 Equilibrium Sorting and Uniqueness

In this section, we show that the equilibria have a clear and simple structure: Section 5.1 shows that every equilibrium exhibits assortative matching if the production function is super/submodular. Section 5.2 considers an additively separable production function (product market) and shows that the equilibrium is unique. Furthermore, these results show that for our second welfare theorem, the efficient allocation is not caught in a widely cast net.

5.1 Assortative Matching

Denote the matching set of skill-i buyers by $M_i = \{j : m_{ij} > 0\} \subseteq J$, this is the set of seller skills with whom buyer i matches. Similarly, for sellers, $M_j = \{i : m_{ij} > 0\} \subseteq I$. The maxima and minima of these sets are denoted $\overline{m}_i = \max M_i$, $\underline{m}_i = \min M_i$, $\overline{m}_j = \max M_j$ and $\underline{m}_j = \min M_j$. We say that a buyer's matching set M_i is convex if $\underline{m}_i < j < \overline{m}_i$ implies that $m_{ij} = 1$ (this is stronger than stating that the matching sets are intervals because it requires that only boundary types can match probabilistically). Convexity is defined analogously for sellers. A matching matrix M exhibits *positive assortative matching* (PAM) if the matching sets are convex and the maxima/minima are weakly increasing. Likewise, M exhibits *negative assortative matching* (NAM) if the matching sets are convex and the maxima/minima are weakly decreasing. Finally, we say that *All Skills Match* if $m_{ij} = 1$ for all i, j.

m_{ij}	j_1	j_2	j_3	j_4	j_5
i_1					
i_2					
i_3					
i_4					
i_5					

Table 3: A PAM matrix: $m_{ij} = 1$ (blue), $0 < m_{ij} < 1$ (green), and $m_{ij} = 0$ (blank)

In Table 3, we depict a matching matrix that satisfies PAM. To maintain PAM, this matrix cannot be modified so that buyer 1 matches with seller 3 (pure or mixed) because that would violate the convexity condition for buyer 1. Likewise, it cannot be that buyer 2 matches with seller 5 because that would violate monotonicity.

The production function G is *supermodular (submodular)* if the marginal productivity of every skill i, $g_{(i+1)j} - g_{ij}$, is strictly increasing (decreasing) in j, and the marginal productivity of every skill j, $g_{i(j+1)} - g_{ij}$, is strictly increasing (decreasing) in i; G is *separable* if the marginal productivity of every skill i is constant in j, and the marginal productivity of every skill j is constant in i.

Previous work established sufficient conditions for positive/negative assortative matching for a single population of agents (Shimer and Smith [2000], Atakan [2006]). However, the single population model is restrictive and does not cover many important settings where there are two different populations, such as labor and product markets. An open question in the literature is whether assortative matching holds when the two populations are not identical.[11] The next result shows that the answers is a firm yes. To our knowledge, this is the first paper which establishes assortativity beyond the single-population framework.

Theorem 2. ***(Sorting)*** *In equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

To outline the argument, we first show that the surplus function s_{ij} inherits super/submodularity from G. We use this observation and Lemma 2 to establish that the bounds of the matching sets are monotonic. We prove convexity from algebraic manipulations of the Constant Surplus equations. In contrast, existing proofs rely heavily on symmetry (Shimer and Smith 2000; Atakan 2006). In the discounting case, to show that the matching sets are convex, Shimer and Smith [2000] place further restriction on the production function which imply that the surplus function s_{ij} is convex[12] whereas our proof works without further restrictions.

Proof. Demonstrating PAM requires demonstrating two components, that the bounds of the matching set are weakly increasing and that the matching set is convex. Throughout, we will use the following key fact: if G is supermodular, then so are the surpluses $[s_{ij}]$.

Increasing Upper Bounds: Fix two buyer skills $i_2 > i_1$. Suppose that $\overline{m}_{i_2} < \overline{m}_{i_1}$. Denote these as $j_2 = \overline{m}_{i_2}$ and $j_1 = \overline{m}_{i_1}$. By IR Matching, it must be that $s_{i_1 j_1} \geq 0 \geq s_{i_2 j_1}$. By supermodularity, then it must be that for every $j < j_1$ it is the case that $s_{i_1 j} > s_{i_2 j}$. This violates the Constant Surplus equations because

$$2c = \sum_{j\in J} y_j m_{i_2 j} s_{i_2 j} = \sum_{j\in M_{i_2}} y_j s_{i_2 j} < \sum_{j\in M_{i_2}} y_j s_{i_1 j} \leq \sum_{j\in M_{i_1}} y_j s_{i_1 j} = \sum_{j\in J} y_j m_{i_1 j} s_{i_1 j} = 2c$$

The case for lower bounds and for submodular G are analogous.

[11]Furthermore, even when the populations are ex-ante symmetric, their investments may be asymmetric and hence the equilibrium will not be symmetric (see Example 2).

[12]In fact, there are examples where G is supermodular and s_{ij} is not convex, and yet there is PAM.

Convexity: Suppose not. That is, there is a buyer i and sellers $j_1 < j < j_2$ such that $m_{ij} < 1$, and $m_{ij_1}, m_{ij_2} > 0$. Then, it must be the case that seller j has a strictly positive surplus with a lower buyer and that buyer is present with non-zero measure. Otherwise

$$2c = \sum_{i'>i, i'\in M_j} x_{i'} s_{i'j} < \sum_{i'>i, i'\in M_j} x_{i'} s_{i'j_2} \leq 2c$$

The middle inequality follows from $s_{i'j_2} \geq s_{ij} + s_{i'j_2} > s_{ij_2} + s_{i'j} \geq s_{ij_2}$ for every $i' > i$ due to the supermodularity of s. Therefore, there is some $i' < i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.

An analogous argument demonstrates that there is:

1. A higher buyer $i' > i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.
2. A lower seller $j' < j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.
3. A higher seller $j' > j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.

Let $\underline{j} = \arg\max_{j' \leq j} s_{ij'}$ and likewise $\overline{j} = \arg\max_{j' \geq j} s_{ij'}$. Similarly, let $\underline{i} = \arg\max_{i' \leq i} s_{i'j}$ and likewise $\overline{i} = \arg\max_{i' \geq i} s_{i'j}$. See below for an illustration of the matching matrix.

	$\dots$	$\underline{j}$	$\dots$	j	$\dots$	$\overline{j}$	$\dots$
$\dots$				0			
$\underline{i}$				1			
$\dots$							
i	0	1		$m_{ij} < 1$		1	0
$\dots$							
$\overline{i}$				1			
$\dots$				0			

Table 4: Convex Matching Sets

Define $y = y_j$, $\underline{y} = \sum_{j'<j, j'\in M_i} y_{j'}$ and $\overline{y} = \sum_{j'>j, j'\in M_i} y_{j'}$. Similarly, $x = x_i$, $\underline{x} = \sum_{i'<i, i'\in M_j} x_{i'}$ and $\overline{x} = \sum_{i'>i, i'\in M_j} x_{i'}$. Notice that $\overline{x}, \underline{x}, \overline{y}, \underline{y} > 0$ as shown above.

By the supermodularity of s, for any $i' > i$, it is the case that $s_{i'\overline{j}} + s_{ij} > s_{i'j} + s_{i\overline{j}}$ and since $s_{ij} \leq 0$, it follows that $s_{i'\overline{j}} > s_{i'j} + s_{i\overline{j}}$. Thus,

$$2c \geq \sum_{i'\geq i, i'\in M_j} x_{i'} s_{i'\overline{j}} > \sum_{i'\geq i, i'\in M_j} x_{i'} (s_{i'j} + s_{i\overline{j}}) = \left(\sum_{i'\geq i, i'\in M_j} x_{i'} s_{i'j} \right) + (x + \overline{x}) s_{i\overline{j}} \tag{16}$$

The strict inequality use the fact that $x_{i'} > 0$ for some $i' > i$.

Next, notice that $s_{\underline{i}j} \geq s_{i'j}$ for all $i' < i$. Therefore,

$$\underline{x} s_{\underline{i}j} = \sum_{i'<i, i'\in M_j} x_{i'} s_{\underline{i}j} \geq \sum_{i'<i, i'\in M_j} x_{i'} s_{i'j} \tag{17}$$

Adding equations (16) and (17) gives

$$2c + \underline{x} s_{\underline{i}j} > \sum_{i'\in M_j} x_{i'} s_{i'j} + (x + \overline{x}) s_{i\overline{j}}$$

And therefore,

$$\underline{x} s_{\underline{i}j} > (x + \overline{x}) s_{i\overline{j}} \tag{18}$$

Similarly:

$$s_{\bar{i}j'} > s_{ij'} + s_{\bar{i}j} \text{ for all } j > j'$$
$$s_{\underline{i}j'} > s_{ij'} + s_{\underline{i}j} \text{ for all } j' > j$$
$$s_{i'\underline{j}} > s_{i'j} + s_{i\underline{j}} \text{ for all } j' < j$$

Repeating the same arguments:
$$\bar{y}s_{i\underline{j}} > (\underline{y} + y)s_{\bar{i}j} \tag{19}$$
$$\bar{y}s_{i\bar{j}} > (y + \underline{y})s_{\underline{i}j} \tag{20}$$
$$\bar{x}s_{\bar{i}j} > (\underline{x} + x)s_{i\underline{j}} \tag{21}$$

As shown earlier, all of the surpluses, $s_{\underline{i}j}, s_{\bar{i}j}, s_{i\underline{j}}, s_{i\bar{j}}$ are positive. Taking the product of Inequalities (18)–(21) and dividing by the surpluses yields:

$$\underline{x}\bar{x}\underline{y}\bar{y} > (\underline{x} + x)(\bar{x} + x)(\underline{y} + y)(\bar{y} + y)$$

which is a contradiction due to the strict inequality.

Separability Implies All Skills Match: By Lemma 2, it is the case that for any two sellers, $w_{j'} - w_j = g_{j'} - g_j$. Therefore, the surplus function is constant $s_{ij'} = g_i + g_{j'} - v_i - w_{j'} = g_i + g_j - v_i - w_j$ and by the Constant Surplus equations, it must be that $s_{ij} = 2c$ for all i, j. So, every pair of agents accept their match. □

Remark 8. The assortative matching result is useful for numerical analysis. For example, in the 5×5 case depicted in Table 3, there are $2^{25} \approx 33.6$ million pure matching matrices, but only 2,762 of them satisfy PAM. In the 5×7 case, there are $2^{35} \approx 34$ trillion pure matching matrices, of which only 21,659 satisfy PAM.[13]

5.2 Uniqueness: Separable Production

We now demonstrate that when the production function is separable, i.e. $g_{ij} = g_i + g_j$, there is a unique equilibrium. To relate to previous work, e.g. Rubinstein and Wolinsky [1985], Gale [1987], we phrase this subsection in the language of a product market. Each seller can produce one unit of a homogeneous good and each buyer desires a single unit. A buyer that invests in skill i receives the payoff α_i from consuming the good and a seller that invests in skill j can produce the good at a cost κ_j. The consumption value α_i is increasing in i and the cost κ_j is decreasing in j. When a buyer and seller meet, their output is $g_{ij} = \alpha_i - \kappa_j$. This production function is separable because the marginal productivity $g_{i'j} - g_{ij}$ is independent of j. As in Gale [1987], we allow for endogenous entry, with outside payoffs equal to u^b for buyers and u^s for sellers. To focus on the interesting case, we ignore the trivial equilibrium where no agent enters, and we assume that there are gains to trade, and so for at least two types, β and σ, $\max_{i \in I, j \in J} g_{ij} - 2c - C^b(i,\beta) - C^s(j,\sigma) > u^b + u^s$ and that not all agents enter, so there are at least two types for which the opposite inequality holds.

[13] At 1000 calculations per second, the difference is between a program taking a millennium and 21 seconds.

Proposition 2. *Any economy with a separable production function (with or without outside options) has a* ***unique*** *equilibrium and its allocation achieves the first best: all skills match and all transactions occur at one price,* $p = \alpha_1 - v_1 - c$.

Theorem 2 demonstrates that with a separable production function, in any equilibrium, All Skills Match. The rest of the proof immediately follows from Lemma 2: since All Skills Match, the marginal values equal marginal productivities, and by separability, $\Delta v_i = \Delta\alpha_i$ and $\Delta w_j = -\Delta\kappa_j$. Thus, the flows and stocks are uniquely pinned down, and the surpluses s_{ij} are constant. As a result, **a law of one price** prevails (all trades occur at one price) and endogenous entry uniquely pins down the price that equates supply and demand. Finally, the agents' private incentives to invest are exactly aligned with the planner, so the equilibrium achieves the first-best. For a formal proof, see Appendix.

6 Robustness

In this section, we extend the baseline model in several directions: Section 6.1 considers asymmetric search costs and bargaining weights, and Section 6.2 considers other CRS meeting functions. We will show that our main results regarding efficient investment, efficient matching, sorting, and existence are robust to modifying the bargaining weights, search costs, and meeting function (satisfying CRS).

6.1 Asymmetric Search Costs and Bargaining Weights

We extend the baseline model by allowing asymmetric search costs and bargaining weights. In every period, each buyer incurs the search cost $c^b > 0$ and each seller incurs the search cost $c^s > 0$. When a buyer with skill i and a seller with skill j accept each other, the buyer receives $v_i + \alpha s_{ij}$ and the seller receives $w_j + (1-\alpha)s_{ij}$.

In the baseline model, $c^b = c^s = c$ and $\alpha = 1-\alpha = 1/2$, and Lemma 1 established that in any equilibrium, the number of buyers B equals the number of sellers S. When the bargaining weights or search costs are asymmetric, the equilibrium state can be unbalanced, $B \neq S$. Recall that in an unbalanced market, the long side of the market is rationed, e.g., if $B < S$, then in each period, every buyer meets a seller with probability 1 and every seller meets a buyer with probability B/S (and vice-versa if $B > S$).

Theorem 3. *For every economy* $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha\rangle$, *let* $r \equiv \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$:

1. *Every equilibrium has the same balance ratio* $r = B/S$.
2. *Given the balance ratio* r, *the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let* $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *maximize total welfare under the previous constraints (7)-(15) and the additional constraint* $r = B/S$. *There are values* (v_i^*), (w_j^*), *and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* $x_i, y_j > 0$.

Proof. Define $\mu = min(B, S)$. In equilibrium, the values satisfy:

$$v_i = (\mu/B)\left(\sum_{j\in J} y_j\left[m_{ij}\left(v_i+\alpha s_{ij}\right)+(1-m_{ij})v_i\right]\right)+\left(1-\mu/B\right)v_i - c^b, \forall i$$

$$w_j = (\mu/S)\left(\sum_{i\in I} x_i\left[m_{ij}\left(w_j+(1-\alpha)s_{ij}\right)+(1-m_{ij})w_j\right]\right)+(1-\mu/S)w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\left(\mu/B\right)}, \forall i \tag{22}$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\left(\mu/S\right)}, \forall j$$

$$\Rightarrow \frac{c^b}{\alpha\left(\mu/B\right)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\left(\mu/S\right)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b} \tag{23}$$

The rest of the proof follows a similar argument as the proof of Theorem 1. For the formal argument, see Appenix . □

Therefore, the search costs and bargaining weights uniquely pin down the balance ratio $r = \frac{c^s}{c^b}\frac{\alpha}{1-\alpha}$. The market is balanced $B = S$ if and only if the bargaining weight equals the search cost ratio $\alpha = \frac{c^b}{c^b+c^s}$. Under any other bargaining weight, the market is imbalanced which is inefficient because one side of the market is rationed. However, imbalance is the only inefficiency: given the balance ratio, there are values that decentralize the efficient investment decisions, search decisions, and steady state as an equilibrium.

Existence and sorting hold as is.

Corollary 3. *An equilibrium exists.*

Theorem 4. *For any economy $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha\rangle$, in equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

The proof of Theorem 4 is essentially the same as the proof of Theorem 2 using the modified constant surplus equations. Finally, the next Proposition states that any economy with asymmetric search costs and bargaining weights has an equivalent economy with symmetric bargaining weights and search costs.

Proposition 3. *Given the economy $\mathscr{E}^{asym}$ with asymmetric search costs c^b and c^s and bargaining weight α, let $\mathscr{E}^{sym}$ denote the same economy only with a symmetric bargaining weight and symmetric search costs $c = \max\{\frac{c^b}{2\alpha}, \frac{c^s}{2(1-\alpha)}\}$. These two economies have the same equilibrium allocations and welfare.*

Proof. Notice that the constant surplus equations of both economies are the same. □

We find the results in this section surprising. A key observation is that if one side of the market has a beneficial position due to a higher bargaining weight or lower search cost, then in equilibrium they will be rationed, and thus effectively incur a higher search cost because they may fail to find a partner in each period. This force effectively equalizes the search costs between the two sides of the market, whereupon the market behaves like our baseline model modulo this imbalance due to rationing.

6.2 Meeting Function

Finally, we consider a general meeting function where $\mu(B,S)$ is the total number of meetings in a period. In every period, each agent can meet at most one other agent, and so $\mu(B,S) \le \min\{B,S\}$. Meetings are still random and the probability that a buyer meets a seller is $\mu(B,S)/B$, while the probability that a seller meets a buyer is $\mu(B,S)/S$. As is standard, we take μ to be homogeneous of degree 1.

Corollary 4. *For every economy* $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha, \mu\rangle$, *let* $r \equiv \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$

1. *Every equilibrium has the same balance ratio* $r = B/S$.
2. *Given the balance ratio* r, *the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let* $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *maximize total welfare under the previous constraints (7)-(15) and the additional constraint* $r = B/S$. *There are values* (v_i^*), (w_j^*), *and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* $x_i, y_j > 0$.

The proof closely follows that of Theorem 1. For the formal argument, see Appendix.

This result clarifies the relationship between our work and Hosios' condition. In every generalized economy, for the state B,S to constitute an equilibrium, it must be that

$$\alpha = \frac{Bc^b}{Bc^b + Sc^s} \qquad \text{(Equilibrium condition)}$$

In the constrained efficient allocation, the states B^*, S^* must satisfy the optimality condition

$$\frac{Bc^b}{Bc^b + Sc^s} = \frac{\partial\mu(B,S)/\partial B}{\mu(B,S)/B} \qquad \text{(Optimality condition)}$$

To decentralize the efficient allocation, it must be that the constrained efficient allocation satisfies the equilibrium condition:

$$\alpha = \frac{B^* c^b}{B^* c^b + S^* c^s} \qquad \text{(Optimal } B^* \text{ substituted into Equilibrium condition)}$$

This yields the usual Hosios condition:

$$\alpha = \frac{\partial\mu(B,S)/\partial B|_{B,S=B^*,S^*}}{\mu(B^*,S^*)/B^*}$$

As previously mentioned, our result and Hosios' [1990] result are complementary. Hosios considers a model with homogeneous agents and shows that there exists a specific bargaining weight that efficiently balances the market. The same applies here. The bargaining weight mechanically pins down the equilibrium balance ratio by $r = \frac{c^s}{c^b}\frac{\alpha}{1-\alpha}$. Our result is about decentralizing the efficient investment and matching decisions and it does not depend on the bargaining weights or search costs.

7 Discussion

This paper developed and analyzed a model where heterogeneous agents acquire skills and then engage in costly search to form productive matches in the market. Despite potential hold-up and matching problems, the main result is that the constrained efficient allocation is an equilibrium of the decentralized market. In addition, regarding the equilibrium structure: we established assortative matching for super/submodular production functions and uniqueness for separable production functions. Importantly, the sorting result applies to two-population models, such as the labor market. Our main results regarding efficient investment, efficient matching, and sorting are robust: they do not depend on the search costs nor the bargaining weights. We mention below several implications and takeaways from our model and results.

Search Externalities: A key tension underlying the efficiency result is that the decisions to acquire skills and to accept or reject potential partners impose externalities on other agents. For instance, if less buyers acquire skill i and more buyers acquire skill $i+1$, then the pool of agents in the market changes: the number of buyers with skill i decreases, the number of buyers with skill $i+1$ increases, and the number of buyers and sellers with other skills may also change because the relative size of their matching partners may increase or decrease. Alternatively, if buyers with skill i accept more partners, then their number in the search pool decreases, and subsequently the number of agents with other skills may increase or decrease since the relative size of their matching partners may change. The planner's solution takes such steady-state search externalities into account. In contrast, in equilibrium, each agent invests and accepts or rejects partners simply by their private incentives, as determined by the value of each skill in the market. In order to achieve the efficient outcome, the equilibrium values must make the agents internalize these externalities.

Discounting: The search literature has considered two types of search costs: *explicit search costs,* which reflect costs people incur per unit of time as they search, and *implicit search costs* due to discounting as payoffs are delayed. When agents discount time, higher skills have higher continuation values and hence higher implicit search costs, which affects the bargaining outcomes. As a result, in equilibrium, the discount factor reduces the incentive to invest, mismatching can occur and sorting may fail for super/submodular production functions (see Shimer and Smith 2000, 2001). By severing the implicit link between values and search costs, our model delivers powerful results: general efficiency and sorting results, existence, and the equilibria have a clear and intuitive structure.

Rubinstein [1982]: Rubinstein's seminal paper studied a bilateral bargaining game with (i) discounting and (ii) explicit time costs. The model with discounting has a unique

SPE that depends smoothly and intuitively on the discount factors, whereas the model with explicit costs has a stark SPE: the player with the smaller cost receives almost the entire pie, and if both players have the same cost, then almost any split of the pie is an SPE. The bargaining literature naturally gravitated towards the discounting model. However, our paper finds the opposite when agents search and bargaining in a market: the explicit search cost model is simple, tractable, and delivers sharp results, whereas the discounting model is more cumbersome to analyze.

Applications: The model is tractable and applicable to various settings. A fundamental question in the labor market is about sorting – when will high-tech firms match with high-skill workers? Theorem 2 establishes a new sorting result in a two-population model. In product markets, match output is typically taken to be the gains from trade, $g_{ij} = u_i - c_j$, i.e. the buyer's utility minus the seller's cost, which is additively separable. Proposition 2 establishes that there is a unique equilibrium and it achieves the first-best allocation. For the marriage market, we should consider a symmetric economy (as in the example in Section 3), and our model may shed new light on questions regarding premarital investments and sorting patterns.

Policy Intervention: In our model, there are two types of inefficiencies. First, there can be multiple equilibria and agents may fail to coordinate on the efficient one. A policy intervention may move the economy away from an inefficient equilibrium. Second, in the case of asymmetric bargaining weights or search costs, the market can be imbalanced, $B \neq S$. In this case, a small search cost subsidy for the short side of the market is generally net beneficial, but a search subsidy targeted at the long side never is (see Proposition 3).

Applications and Simulations: The welfare and sorting results are useful for computational analyses. In particular, solving the planner's problem solely requires finding an allocation whereas solving for an equilibrium also requires finding values . Notice that for an n-skill economy, the endogenous variables $N, (x_i), (y_j), (\beta_i), (\sigma_j)$ are of order n, but the matching matrix $[m_{ij}]$ is of order n^2. The assortative matching result reduces the number of matching variables from n^2 to $2n$, which brings the dimensionality of the whole problem from $O(n^2)$ to $O(n)$. A further advantage of the welfare theorem is that seeing the economy through the planner's lens may provide intuition that is not evident from the equilibrium conditions. Calibration of the model to fit empirical data lies beyond the scope of the current paper, but the theoretical results found here offer promise.

References

Daron Acemoglu. A microfoundation for social increasing returns in human capital accumulation. *The Quarterly Journal of Economics*, 111(3):779–804, 1996.

Daron Acemoglu and Robert Shimer. Holdups and efficiency with search frictions. *International Economic Review*, 40(4):827–849, 1999.

Yair Antler, Daniel Bird, and Daniel Fershtman. Learning in the marriage market: The economics of dating. *Working Paper*, 2025.

Alp E. Atakan. Assortative matching with explicit search costs. *Econometrica*, 74(3):667–680, 2006.

Ken Burdett and Melvyn G. Coles. Transplants and implants: The economics of self-improvement. *International Economic Review*, 42(3):597–616, 2001.

Kenneth Burdett and Melvyn G. Coles. Long-term partnership formation: Marriage and employment. *The Economic Journal*, 109(456):307–334, 1999.

Xiaoming Cai, Pieter Gautier, and Ronald Wolthoff. Search, screening, and sorting. *American Economic Journal: Macroeconomics*, 17(3):205–236, 2025.

Hector Chade and Ilse Lindenlaub. Risky matching. *The Review of Economic Studies*, 89 (2):626–665, 2022.

Harold L. Cole, George J. Mailath, and Andrew Postlewaite. Efficient non-contractible investments in large economies. *Journal of Economic Theory*, 101(2):333 – 373, 2001.

Peter A. Diamond. Wage determination and efficiency in search equilibrium. *The Review of Economic Studies*, 49(2):217–227, 1982.

Deniz Dizdar. Two-sided investment and matching with multidimensional cost types and attributes. *American Economic Journal: Microeconomics*, 10(3):86–123, August 2018.

Jan Eeckhout and Philipp Kircher. Sorting and decentralized price competition. *Econometrica*, 78(2):539–574, 2010.

Matthew Elliott and Francesco Nava. Decentralized bargaining in matching markets: Efficient stationary equilibria and the core. *Theoretical Economics*, 14(1):211–251, 2019.

Matthew Elliott and Eduard Talamàs. Investment in matching markets. *Online and Matching-Based Market Design*, 448, 2023.

Douglas Gale. Limit theorems for markets with sequential bargaining. *Journal of Economic Theory*, 43(1):20 – 54, 1987.

Arthur J. Hosios. On the efficiency of matching and related models of search and unemployment. *The Review of Economic Studies*, 57(2):279–298, 1990.

Robert Janin. Directional derivative of the marginal function in nonlinear programming. In *Sensitivity, Stability and Parametric Analysis*, pages 110–126. Springer, 1984.

Belen Jerez. Competitive search equilibrium with multidimensional heterogeneity and two-sided ex-ante investments. *Journal of Economic Theory*, 172:202 – 219, 2017.

Stephan Lauermann. Dynamic matching and bargaining games: A general approach. *American Economic Review*, 103(2):663–89, 2013.

Stephan Lauermann, Georg Nöldeke, and Thomas Tröger. The balance condition in search-and-matching models. *Econometrica*, 88(2):595–618, 2020.

George J. Mailath, Andrew Postlewaite, and Larry Samuelson. Pricing and investments in matching markets. *Theoretical Economics*, 8(2):535–590, 2013.

Mihai Manea. Steady states in matching and bargaining. *Journal of Economic Theory*, 167:206 – 228, 2017.

Adrian M. Masters. Efficiency of investment in human and physical capital in a model of bilateral search and bargaining. *International Economic Review*, 39(2):477–494, 1998.

Dale T. Mortensen. Property rights and efficiency in mating, racing, and related games. *The American Economic Review*, 72(5):968–979, 1982a.

Dale T Mortensen. The matching process as a noncooperative bargaining game. In *The Economics of Information and Uncertainty*, pages 233–258. University of Chicago Press, 1982b.

Dale T Mortensen and Randall Wright. Competitive pricing and efficiency in search equilibrium. *International Economic Review*, 43(1):1–20, 2002.

Georg Noldeke and Larry Samuelson. Investment and competitive matching. *Econometrica*, 83(3):835–896, 2015.

Christopher A. Pissarides. Short-run equilibrium dynamics of unemployment, vacancies, and real wages. *The American Economic Review*, 75(4):676–690, 1985.

Ariel Rubinstein. Perfect equilibrium in a bargaining model. *Econometrica: Journal of the Econometric Society*, pages 97–109, 1982.

Ariel Rubinstein and Asher Wolinsky. Equilibrium in a market with sequential bargaining. *Econometrica: Journal of the Econometric Society*, pages 1133–1150, 1985.

Lloyd S Shapley and Martin Shubik. The assignment game i: The core. *International Journal of game theory*, 1(1):111–130, 1971.

Shouyong Shi. Frictional assignment i efficiency. *Journal of Economic Theory*, 98(2):232 – 260, 2001.

Robert Shimer. The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy*, 113(5):996–1025, 2005.

Robert Shimer and Lones Smith. Assortative matching and search. *Econometrica*, 68(2): 343–369, 2000.

Robert Shimer and Lones Smith. Matching, search, and heterogeneity. *The BE Journal of Macroeconomics*, 1(1), 2001.

8 Appendix

8.1 Remaining Proofs for Theorem 1:

We first prove the non-interior case and then the constant rank constraint qualification.

Proof. z **is non-interior:**

Given any optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$, the FOCs imply that there are shadow values (v_i), (w_j) such that (see proof of Theorem 1 in text):

$$\sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) \geq 2c \text{ with equality when } x_i > 0$$

$$\sum_i x_i m_{ij}\left(g_{ij} - v_i - w_j\right) \geq 2c \text{ with equality when } y_j > 0$$

$$N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

where $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

The above equations demonstrate the Constant Surplus equations for all i where $x_i > 0$. But, the Constant Surplus equation may not hold for skills i where $x_i = 0$. Therefore, for any skill i where $x_i = 0$, we define v_i^* to be the unique value which solves $\sum_j y_j \max\{g_{ij} - v_i^* - w_j, 0\} = 2c$. For any skill i where $x_i > 0$, we define $v_i^* = v_i$. Likewise, for sellers j where $y_j = 0$, define w_j^* to be the unique value which solves $\sum_i x_i \max\{g_{ij} - v_i - w_j^*, 0\} = 2c$. For sellers j where $y_j > 0$, define $w_j^* = w_j$. Define a matching matrix by $m_{ij}^* = \mathbf{1}_{g_{ij} - v_i^* - w_j^* > 0}$ whenever $x_i = 0$ or $y_j = 0$ and setting $m_{ij}^* = m_{ij}$ otherwise.

It now remains to be seen that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ satisfies the equilibrium constraints. **The Constant Surplus Equations hold**: For any skill i where $x_i > 0$, from the above, we have that $\sum_j y_j m_{ij}^*\left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = 2c$ because $v_i^* = v_i$ and whenever $y_j > 0$, then $m_{ij} = m_{ij}^*$ and $w_j = w_j^*$. For any skill i where $x_i = 0$,

$$\sum_j y_j m_{ij}^*\left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j \max\left(g_{ij} - v_i^* - w_j^*, 0\right)$$
$$= \sum_j y_j \max\left(g_{ij} - v_i^* - w_j, 0\right) = 2c$$

because $w_j^* = w_j$ whenever $y_j > 0$. The same argument demonstrates the Constant Surplus equations for the sellers.

IR Matching holds: For any two skills i, j where $x_i = 0$ or $y_j = 0$, the IR Matching condition holds by definition. For any two skills i, j where $x_i > 0$ and $y_j > 0$, then $v_i^* = v_i$, $w_j^* = w_j$, and $m_{ij}^* = m_{ij}$ and the IR Matching condition is a direct consequence of FOC(m_{ij}).

Optimal Investments: Regarding optimal investments, just as in the proof in the main section, here the values (v_i) satisfy incentive compatibility for investments. However, it is not readily evident that the values (v_i^*) satisfy incentive compatibility because the values for unrealized skills are modified, and may be increased. We now show that for all unrealized skills $v_i \geq v_i^*$.

Since $m_{ij}x_iy_j = m^*_{ij}x_iy_j$ for any two skills i, j, the policy $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$ is admissible and optimal. By the constraint qualifications, there are values $(\hat{v}_i)$, $(\hat{w}_j)$ which satisfy the FOCs for $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$. From FOC($\beta_i$), we have that the marginal values are equal for all i, $\hat{v}_i - \hat{v}_{i-1} = C^b(i, \beta_i) - C^b(i-1, \beta_i) = v_i - v_{i-1}$. Likewise, for all sellers j, $\hat{w}_j - \hat{w}_{j-1} = w_j - w_{j-1}$. Thus, there is a constant t such that $\hat{v}_i + \hat{w}_j = v_i + w_j + t$ for all i, j. For any skill i such that $x_i > 0$,

$$2c = \sum_j y_j m^*_{ij}(g_{ij} - \hat{v}_i - \hat{w}_j) = \sum_j y_j m^*_{ij}(g_{ij} - v_i - w_j - t)$$
$$= \sum_j y_j m_{ij}(g_{ij} - v_i - w_j - t) = 2c - t\sum_{ij} y_j m_{ij}$$

Therefore, $t = 0$ and so $\hat{v}_i + \hat{w}_j = v_i + w_j$ for all i, j.
For any unchosen skill i,

$$\sum_j y_j m^*_{ij}\left(g_{ij} - v_i^* - w_j\right) = 2c \geq \sum_j y_j m^*_{ij}\left(g_{ij} - \hat{v}_i - \hat{w}_j\right) = \sum_j y_j m^*_{ij}\left(g_{ij} - v_i - w_j\right)$$

Therefore, we can conclude that $v_i \geq v_i^*$. This demonstrates incentive compatibility. For every skill i, $v_i \geq v_i^*$ with equality if $x_i > 0$. As (v_i) satisfied incentive compatibility and (v_i^*) differs by only lowering the value of unrealized skills, the values (v_i^*) also satisfy incentive compatibility. This establishes that for the values $(v_i^*), (w_j^*)$, no agent wishes to choose any unchosen skill and completes the proof. □

Constraint Qualification

Lemma 3. *The planner's optimization problem satisfies the Constant Rank Constraint Qualification.*

Proof. We show that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in a vicinity of the optimal point is constant (Janin [1984]).
There is an immediate linear dependency among the gradients:

$$\sum_{i \in I} \alpha \nabla flow_i^b - \sum_{j \in J} \alpha \nabla flow_j^s = 0$$

which follows from

$$\sum_{i \in I} flow_i^b - \sum_{j \in J} flow_j^s = 0$$

We will show that this is the only linear dependency, which suffices for the constant rank constraint qualification. Suppose that $\sum_n \alpha_n \nabla_n = 0$ where the summation is over all the active gradients. To simplify notation, we label the skills as $I = \{0, \dots, k\}$ and $J = \{0, \dots, l\}$. Notice first that (β_i) and (σ_j) appear only in the flow constraints:

∇	β_1	β_2	β_3	$\dots$	β_k	N	$\sigma_j, x_i, y_j, m_{ij}$
$\nabla flow_0^b$	$-f^b(\beta_1)$	0	0	0	0	$-x_0 \sum_{j\in J} y_j m_{0j}$	$\dots$
$\nabla flow_1^b$	$f^b(\beta_1)$	$-f^b(\beta_2)$	0	0	0	$-x_1 \sum_{j\in J} y_j m_{1j}$	$\dots$
$\nabla flow_2^b$	0	$f^b(\beta_2)$	$-f^b(\beta_3)$	0	0	$-x_2 \sum_{j\in J} y_j m_{2j}$	$\dots$
$\dots$	0	0	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\nabla flow_{k-1}^b$	0	0	0	$f^b(\beta_{k-1})$	$-f^b(\beta_k)$	$-x_{k-1} \sum_{j\in J} y_j m_{k-1,j}$	$\dots$
$\nabla flow_k^b$	0	0	0	0	$f^b(\beta_k)$	$-x_k \sum_{j\in J} y_j m_{k,j}$	$\dots$

Since β_i only shows up in up in $flow_i^b$, $flow_{i-1}^b$ it must be that

$$0 = \sum_n \alpha_n \frac{\partial f_n}{\partial \beta_{i'}} = \sum_{i\in I} \alpha_i \frac{\partial flow_i^b}{\partial \beta_{i'}} = f^b(\beta_{i'})\alpha_{i'} - f^b(\beta_{i'})\alpha_{i'-1} \text{ for all } i'$$

Thus, there is an α such that $\alpha_i = \alpha$ for all the coefficients of the constraints $\nabla flow_i^b$. Similarly, there is a χ so that $\alpha_j = \chi$ for all the coefficients of the constraints $\nabla flow_j^s$. Furthermore, N only shows up in the flow constraints, so it must be that

$$-\alpha \sum_i x_i \sum_j y_j m_{ij} - \chi \sum_j y_j \sum_i x_i m_{ij} = 0$$

which implies $\chi = -\alpha$ (notice that $\sum_i x_i \sum_j y_j m_{ij} = 1/N$). Therefore, there is exactly one linear dependency

$$\sum \alpha_i \nabla flow_i^b + \sum_j \alpha_j \nabla flow_j^s = \alpha \left(\sum_i \nabla flow_i^b - \sum_j \nabla flow_j^s \right) = 0$$

Second, the coefficients on $\nabla(x_i \geq 0)$ and ∇X are all zeros. The reason is that x_i appears in the flow constraints and the constraints $x_i \geq 0$ and $X = 0$. By the previous step, in any linear dependence, the flow constraints cancel each other out, so only the constraints $x_i \geq 0$ and $X = 0$ are relevant. Therefore, if $\sum_i \xi_i \nabla(x_i \geq 0) + \xi \nabla X = 0$, then $0 = \xi_i \frac{\partial x_i}{\partial x_i} + \xi \frac{\partial X}{\partial x_i} = \xi_i - \xi$, and so $\xi_i = \xi$ for all i. If $\xi \neq 0$, then it must be that every inequality on x is active, so $x_i = 0$ for every i, contradicting $0 = X = 1 - \sum_i x_i$, which holds in any admissible tuple. The same argument applies to the y_j. So $\xi_i = \xi = \xi_j = 0$ for all i, j.

Third, the coefficients on the m_{ij} constraints are zeros. The reason is that the variable m_{ij} appears only in the flow equations and the inequality constraints on m_{ij}. The flow constraints cancel each other out. For the m_{ij} constraints, $\nabla(1 \geq m_{ij} \geq 0) = (0, \dots 0, \pm 1, 0 \dots)$ and at most one of the m_{ij} constraints can be active where the only non-zero element is in the m_{ij} coordinate and so these gradients coefficients must be 0. □

8.2 Proof of Proposition 1:

Proof. Consider the economies $\Gamma_c = \langle F^b, F^s, I, J, C^b, C^s, G, c \rangle$ indexed by their search cost c and denote its constrained efficient welfare as $\mathcal{W}_c$. Denote an optimal allocation as x_c with associated population N_c (there may be multiple optimal allocations). Notice that by an imitation argument, $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c' - c)$ because the planner could implement $x_{c'}$ when faced with the economy x_c. This implies that welfare is decreasing in c, as expected. Reversing c and c' gives $2N(c)(c' - c) + \mathcal{W}_{c'} \geq \mathcal{W}_c$. Taking $c' > c$. this implies that $|\mathcal{W}_c - \mathcal{W}_{c'}| \leq 2N(c)(c' - c)$. That is, when $N(c)$ is unique, it is the case that $\frac{\partial \mathcal{W}_c}{\partial c} = -2N(c)$.

Otherwise, $N(c)$ can take a set of values, and the left-derivative is $-2\sup N(c)$ and the right-derivative is $-2\inf N(c)$. To see convexity of $\mathscr{W}_c$, it suffices to demonstrate that N is non-increasing in c. Take $c' > c$. Since $\mathscr{W}_c \geq \mathscr{W}_{c'} + 2N(c')(c'-c)$, and similarly $\mathscr{W}_{c'} \geq \mathscr{W}_c + 2N(c)(c-c')$. Adding these two equations together gives $0 > 2(N(c') - N(c))(c'-c)$ and therefore $N(c) \geq N(c')$. □

8.3 Proof of Proposition 2:

Proof. The first-best allocation is unique and satisfies:

First-Best Matching: All pairs match. Since the marginal productivity of an agent is not affected by the skills of her partner, all pairs match to minimize the search cost.

First-Best Investment: Buyer β and seller σ acquire the skills: $i^*(\beta) = \arg\max_i \alpha_i - C^b(i,\beta)$ and $j^*(\sigma) = \arg\max_j -\kappa_j - C^s(j,\sigma)$. Denote by $C^{b*}(\beta) = C^b(i^*(\beta),\beta)$ the investment cost buyer β pays to acquire the efficient skill, and likewise $C^{s*}(\sigma) = C^s(j^*(\sigma),\sigma)$.

The social welfare of a match between buyer β and seller σ is $\omega(\beta,\sigma) = \alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c$. The assumption before the proof implies that there are types, $\beta', \sigma', \hat{\beta}, \hat{\sigma}$ such that $\omega(\beta',\sigma') > u^b + u^s > \omega(\hat{\beta},\hat{\sigma})$. So, in the first-best, some agents enter and others don't.[14]

First-Best Entry: Buyer β and seller σ enter iff $\beta \leq \beta_0$ and $\sigma \leq \sigma_0$. The entry thresholds are pinned down by[15] $F^b(\beta_0) = F^s(\sigma_0)$ and $\omega(\beta_0,\sigma_0) = u^b + u^s$.

Since g is separable, Lemma 2 implies that in equilibrium, the marginal value equal the marginal productivity: $\Delta v_i = \alpha_{i+1} - \alpha_i$, for every i, and $\Delta w_j = -(\kappa_{j+1} - \kappa_j)$, for every j. Therefore, the match surplus $s_{ij} = \alpha_i - \kappa_j - v_i - w_j$ is constant. As a result:

Equilibrium Matching: Theorem 2 demonstrates that in every equilibrium, all skills match.

Equilibrium Investment: The individually optimal investments satisfy

$$\arg\max_i \left\{ v_i - C^b(i,\beta) \right\} = \arg\max_i \left\{ \alpha_i - C^b(i,\beta) \right\}, \text{ for evey } \beta$$
$$\arg\max_j \left\{ w_j - C^s(j,\sigma) \right\} = \arg\max_j \left\{ -\kappa_j - C^s(j,\sigma) \right\}, \text{ for every } \sigma$$

The maximizers are equal because $\alpha_i - v_i$ and $-\kappa_j - w_j$ are constant

Equilibrium Entry: First, we show that there is entry. If not, then $v_{i^*(\beta)} - C^{b*}(\beta) \leq u^b$ and $w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^s$, for all β, σ, and so $v_{i^*(\beta)} - C^{b*}(\beta) + w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^b + u^s$. Substituting in the Constant Surplus equations, it follows that, $\alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c \leq u^b + u^s$, which violates the assumption that there are types, β', σ' such that $\omega(\beta',\sigma') > u^b + u^s$. By a similar argument, it cannot be that all agents enter. Second, since some agents enter and others do not, denote by $\underline{\beta}, \underline{\sigma}$ the threshold types for whom

[14] The case where everyone enters is trivial.

[15] Since buyers and sellers exit in equal numbers, in a steady state they must also enter in equal numbers.

the entry constraints hold with equality, notice that

$$u^b + u^s = v_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) + w_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right)$$
$$= \alpha_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) - \kappa_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) - 2c = \omega(\underline{\beta},\underline{\sigma})$$

The second equality follows from the Constant Surplus equation, $v_i + w_j = \alpha_i - \kappa_j - 2c$. In a steady state, the same measure of buyers and sellers enter, $F^b(\underline{\beta}) = F^s(\underline{\sigma})$. These two equations are the same as the equations that characterized the first-best entry decisions, and therefore it must be that $\underline{\beta} = \beta_0$ and $\underline{\sigma} = \sigma_0$. □

8.4 Proof of Theorem 3 and Corollaries 2 and 4

This subsection proves Theorem 3 and Corollaries 2 and 4. We extend the baseline model to the generalized economy $\mathscr{E} = \langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha, \mu, u^b, u^s\rangle$ by adding the following additional features:

- Asymmetric search costs c^b and c^s and bargaining weight α (as in Section 6.1).
- A meeting function $\mu(B,S)$ specifying the total number of meetings in each period and satisfying constant returns to scale (as in Section 6.2).
- Agents have outside options u^b and u^s and so entry is endogenous (as in Section 4.1). To avoid trivial outcomes, we maintain the assumption that there are gains to trade for at least two types, β and σ, so that

$$\max_{i\in I, j\in J} \mu(1,1) g_{ij} - c^b - c^s - C^b(i,\beta) - C^s(j,\sigma) > u^b + u^s$$

We will now prove a more general version of the the previous results.

Corollary. *For every generalized economy, let* $r \equiv \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$*:*

1. *Every equilibrium has the same balance ratio* $\frac{B}{S} = r$.
2. *Given the balance ratio* r*, the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let* $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *maximize total welfare under the previous constraints (7)-(15) and the additional constraint* $\frac{B}{S} = r$*. There are values* (v_i^*)*,* (w_j^*)*, and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m^*_{ij} = m_{ij}$ *for all* i, j *such that* $x_i, y_j > 0$.

Proof. 1) Let $\mu = \mu(B,S)$. As we previously showed in Section 6.1, in equilibrium, the values satisfy:

$$v_i = \left(\mu/B\right)\left(\sum_{j\in J} y_j \left[m_{ij}\left(v_i + \alpha s_{ij}\right) + (1-m_{ij}) v_i\right]\right) + \left(1 - \mu/B\right) v_i - c^b, \forall i$$

$$w_j = \left(\mu/S\right)\left(\sum_{i\in I} x_i \left[m_{ij}\left(w_j + (1-\alpha) s_{ij}\right) + (1-m_{ij}) w_j\right]\right) + (1-\mu/S) w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\left(\mu/B\right)}, \forall i \tag{24}$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\left(\mu/S\right)}, \forall j$$

$$\Rightarrow \frac{c^b}{\alpha\left(\mu/B\right)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\left(\mu/S\right)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b} \tag{25}$$

2) Decentralizing the efficient allocation given r. To simplify, we focus on the case where the state is interior and the proof repeats that argument with the appropriate modifications. The same could be done for the boundary case as well. The original planner's problem 6 is modified because the agents have an outside option and there is a general meeting function, and so the measure of buyers B need not equal the measure of sellers S. The planner now chooses the state $z = \left(B, S, (x_i), (y_j)\right)$ instead of $z = \left(N, (x_i), (y_j)\right)$, the investment thresholds, and the matching rule to maximize

$$\mathscr{W} = \mu(B,S)\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta)d\beta$$
$$-\sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma)d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma$$

subject to the steady state conditions,

$$flow_i = \int_{\beta_{i+1}}^{\beta_i} f^b(\beta)d\beta - x_i\mu(B,S)\sum_{j\in J} y_j m_{ij} = 0, \forall i$$

$$flow_j = \int_{\sigma_{j+1}}^{\sigma_j} f^s(\sigma)d\sigma - y_j\mu(B,S)\sum_{i\in I} x_i m_{ij} = 0, \forall j$$

$$B, S \geq 0$$

$$x_i \geq 0, \forall i$$

$$y_j \geq 0, \forall j$$

$$X = 1 - \sum_{i\in I} x_i = 0$$

$$Y = 1 - \sum_{j\in J} y_j = 0$$

$$1 \geq m_{ij} \geq 0, \forall i, j$$

$$F^b(\beta_{|I|}) = F^s(\sigma_{|J|}) = 0$$

$$B - rS = 0$$

Notice that taking weighted sums of the flow conditions implies that $F^b(\beta_0) = F^s(\sigma_0)$. The planner's problem is modified in four ways: i) agents can take an outside option which is included in the objective function and the conditions $F(\beta_0) = 1$ and $F(\sigma_0) = 1$ are removed; ii) the measure of buyers B and sellers S may differ and since we assumed that the are gains to trade, the conditions $B, S \geq 0$ will not bind at the efficient solution; iii) we add the balance ratio constraint $\frac{B}{S} = r$; and iv) the Inflow=Outflow equations are modified because the outflow of buyers and sellers is

$$(Bx_i)\left(\frac{\mu(B,S)}{B}\right)\sum_{j\in J} y_j m_{ij} = x_i\mu(B,S)\sum_{j\in J} y_j m_{ij}, \forall i$$
$$\left(Sy_j\right)\left(\frac{\mu(B,S)}{S}\right)\sum_{i\in I} x_i m_{ij} = y_j\mu(B,S)\sum_{i\in I} x_i m_{ij}, \forall j$$

The KKT regularity conditions continue to hold, by the same arguments as in Theorem 1 (because the linear dependencies of the gradients do not change).

Replacing $B = rS$ in the objective:

$$\begin{aligned}
\mathscr{W} &= \mu(rS,S)\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - rSc^b - Sc^s - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta)d\beta \\
&- \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma)d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma \\
&= S\mu(r,1)\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - rSc^b - Sc^s - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta)d\beta \\
&- \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma)d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma
\end{aligned}$$

As in the proof of Theorem 1, we define the Lagrangian and taking the FOC we get:

$$\mu(r,1)\left(\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij}\left(g_{ij} - v_i - w_j\right)\right) - rc^b - c^s = 0$$

(Recall that $S > 0$ and so the multiplies on this constraint is 0).

So,

$$\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} s_{ij} = \frac{rc^b + c^s}{\mu(r,1)}$$

FOC($\mathbf{x_i}$): $\mu(rS,S)\sum_{j\in J} y_j m_{ij} g_{ij} - v_i\mu(rS,S)\sum_{j\in J} y_j m_{ij} - \mu(rS,S)\sum_{j\in J} w_j y_j m_{ij} - \gamma - \phi_i = 0$

where $\phi_i x_i = 0$. Analogously to the proof of Theorem 1:

$$S\mu(r,1)\sum_j y_j m_{ij} s_{ij} = \gamma + \phi_i \tag{26}$$

Multiplying by x_i and summing, $\gamma = S\mu(r,1)\sum_{i\in I}\sum_{j\in J} x_i m_{ij} y_j s_{ij}$

Substituting in from $FOC(S)$:

$$\gamma = S\mu(r,1)\frac{rc^b + c^s}{\mu(r,1)} = S(rc^b + c^s) = Bc^b + Sc^s$$

Therefore, from equation (26) we get that

$$\sum_j y_j m_{ij} s_{ij} = \frac{Bc^b + Sc^s}{\mu(rS,S)} = \frac{Bc^b + Sc^s}{\mu(B,S)} \tag{27}$$

Likewise:

$$\textbf{FOC}(\mathbf{y_i}): \quad \mu(rS,S)\sum_{i\in I} x_i m_{ij} g_{ij} - w_j \mu(rS,S)\sum_{i\in I} x_i m_{ij} - \mu(rS,S)\sum_{i\in I} v_i x_i m_{ij} - \eta - \psi_j = 0$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{\lambda + \psi_j}{\mu(rS,S)}$$

Using the previous two equations and multiplying by y_j and then summing, we obtain:

$$\frac{Bc^b + Sc^s}{\mu(rS,S)} = \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} s_{ij} = \frac{\lambda}{\mu(rS,S)}$$

Therefore, for interior allocations, $\psi_j = 0$ and:

$$\sum_i x_i m_{ij} s_{ij} = \frac{Bc^b + Sc^s}{\mu(rS,S)} = \frac{Bc^b + Sc^s}{\mu(B,S)} \tag{28}$$

To decentralize the optimal allocation, we show that the shadow values of the flow constraints $(v_i),(w_j)$ together with the matching matrix M and state z constitute an equilibrium. Notice that the balance ratio $\frac{B}{S} = r \equiv \frac{\alpha c^s}{(1-\alpha)c^b} \iff \alpha = \frac{Bc^b}{Bc^b + Sc^s}$

We first show that the equilibrium constant surplus equations 24 hold, that is,

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{Bc^b}{\alpha\mu(B,S)}, \forall i$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{Sc^s}{(1-\alpha)\,\mu(B,S)}, \forall j$$

Notice that these equations coincide with equations 27 and 28 whenever $\alpha = \frac{Bc^b}{Bc^b + Sc^s}$.

The FOC(β_0) condition is precisely the equilibrium entry condition for buyers, $v_0 - C^b(0,\beta_0) = u^b$, that is, the shadow value v_0 makes the threshold type β_0 indifferent. Likewise, the FOC(σ_0) condition is precisely the equilibrium entry condition for sellers.

The rest of the proof uses the same argument as in Theorem 1: the FOC[m_{ij}] and the complementary slackness conditions imply that the values and matching matrix satisfy the equilibrium matching conditions $s_{ij} > 0 \to m_{ij} = 1$ and $s_{ij} < 0 \to m_{ij} = 0$; and FOC[$\beta_i$] and FOC[$\sigma_j$] imply that the constrained efficient investments are incentive compatible. □