Benefits and Challenges of Ambiguous Product Information*

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Abstract

We study the welfare effects of ambiguous product information for a buyer with α-max-min preferences and a price-setting seller. The buyer privately receives information about her valuation. We show that both the seller and the buyer can benefit when this information is ambiguous, and we characterize all possible combinations of producer and consumer surplus, as evaluated under ambiguity-sensitive preferences. Ambiguity concerning the valuation perceived by the buyer when making the purchase decision can induce the seller to change the price. Before receiving information, ambiguity concerning the purchase decision can make the buyer optimistic about buying only for high valuations, which relaxes the participation constraint.

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1 Introduction

Information from a variety of sources may affect consumers' purchase decisions. Common sources of information include product packaging and labeling, online reviews, company websites, advertising, social media platforms, word of mouth, and experiences gained during a trial period. Some sources typically provide clear and accurate information, such as labeling that is subject to regulation or reviews by trusted experts. Other sources, however, may provide more obscure information and, most importantly, their reliability may be difficult to assess by consumers. For example, customer reviews on e-commerce platforms could be fake or manipulated, and the biases in recommendations by influencers on social media may not easily be recognized. As a result, consumers often decide whether to buy a product based on information that is to some extent ambiguous.

In this paper, we study the welfare effects of ambiguous information.[1] We consider an ambiguity-sensitive buyer who learns arbitrarily ambiguous information about how well her preferences match the product offered by a price-setting seller. We find that ambiguity is not necessarily harmful: the right kind of ambiguous information makes the buyer or the seller better off than *every* possible unambiguous information.

In our model, the seller offers an object for which the buyer's valuation is uncertain. Initially, the seller and the buyer have a common prior about the buyer's valuation. The buyer privately learns additional information by observing a signal drawn from a signal structure. The seller sets the price knowing the signal structure (but not the signal), and the buyer decides whether to buy. To introduce ambiguity, we assume there is a payoff-irrelevant state that the buyer perceives as ambiguous. That is, she has multiple priors about this state that she deems possible. The signal structure draws a signal that can be arbitrarily correlated with both the valuation and the payoff-irrelevant state. Thus, upon observing the signal, the buyer may perceive ambiguity about her valuation. We assume the buyer has α-max-min preferences: she puts weight $\alpha \in [0,1]$ on her expected utility under the most pessimistic belief and weight $1-\alpha$ on her expected utility under the most optimistic belief.[2] The parameter α represents the buyer's attitude towards ambiguity and allows for varying the degree of ambiguity aversion. The seller

[1] Ambiguity refers to uncertainty about probabilities as opposed to risk. Ellsberg (1961) established that the distinction between risk and ambiguity is behaviorally meaningful.

[2] This representation of preferences under ambiguity has been proposed by Hurwicz (1951). See, e.g., Ghirardato et al. (2004) for details. The special case $\alpha = 1$ corresponds to max-min preferences (Gilboa and Schmeidler, 1989).

is ambiguity neutral, having a unique prior about the payoff-irrelevant state. A signal structure is unambiguous if the signal is independent of the payoff-irrelevant state.

Ambiguity affects the buyer at two points in time. At the interim stage when observing the signal, the buyer infers ambiguous information about her (ex post) valuation, which determines her (interim) willingness to pay and thus her purchase decision. In turn, the distribution of willingness to pay from the seller's perspective (i.e., the demand he faces) determines his pricing decision. At the ex ante stage before observing the signal, the buyer has an unambiguous prior about her valuation, but she perceives ambiguity about her future purchase decision. The buyer's ex ante α-max-min expected utility measures the consumer surplus in our welfare analysis. The seller's expected revenue measures producer surplus.

Our first main result is that both sides can benefit from ambiguity. We first focus on the buyer. We show that there are ambiguous signal structures under which the buyer is strictly better off from the ex ante perspective than under any unambiguous signal structure. Importantly, we uncover two intuitive effects that are beneficial for the buyer: interim pessimism concerning ambiguity about the valuation and ex ante optimism concerning ambiguity about the purchase decision.

Interim pessimism occurs when ambiguity lowers the buyer's willingness to pay and, in turn, the reduced demand prompts the seller to set a lower price than under any unambiguous information. The willingness to pay decreases because signals can imply in some payoff-irrelevant states that the valuation is very low. The lower price directly benefits the buyer, as signal structures can induce interim pessimism without exposing the buyer to ambiguity at the ex ante stage about the purchase decision.[3] It is noteworthy that this effect does not require the buyer to be strongly ambiguity averse: it is feasible for all positive α.

Ex ante optimism occurs when some signals provide very accurate information about the valuation but these signals only realize in payoff-irrelevant states that the seller deems unlikely. Thus, the seller's pricing decision is unaffected. Yet, these signals are beneficial for the buyer ex ante whenever $\alpha < 1$: under her most optimistic beliefs these signals may realize and result in the best possible purchase decision (i.e., buying if and only if the valuation is above the price).

We now turn to the seller. Whenever $\alpha < 1$, there are ambiguous signal structures under which the seller is strictly better off than under any unambiguous

[3]Thus, in our setting, the ex post payoffs of a buyer who is ambiguity averse in the sense of Gilboa and Schmeidler (1989) can be higher in every state than the ex post payoffs of a buyer who is ambiguity neutral.

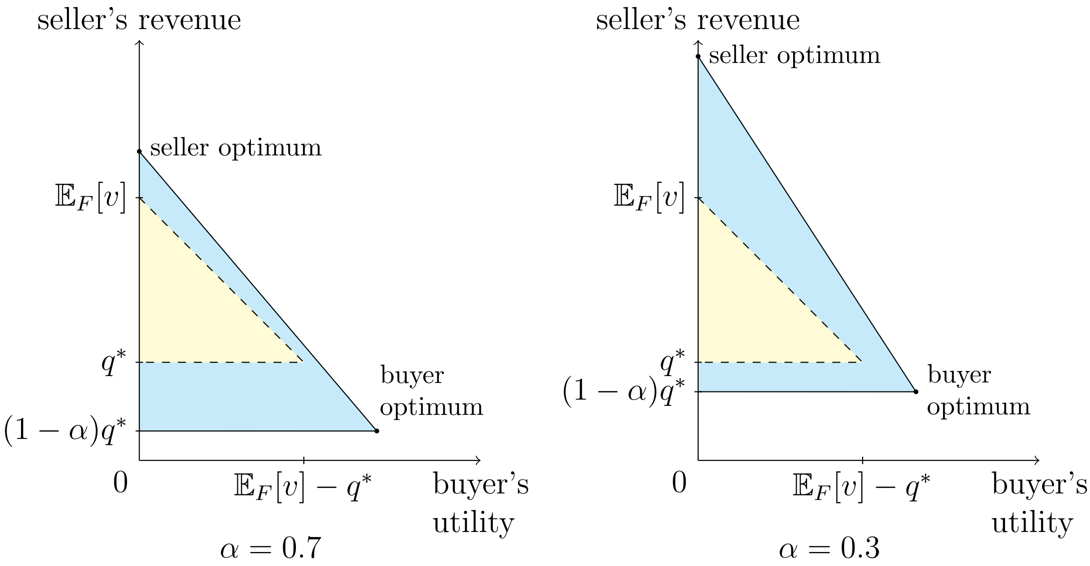

Figure 1: All outcomes that can obtain with ambiguous information. The yellow subset can obtain with unambiguous information, yielding revenue of at least q^*. $\mathbb{E}_F[v]$ is the expected valuation under the common prior.

signal structure. These signal structures combine ex ante optimism with interim optimism, which is a counterpart to interim pessimism. Interim optimism occurs when ambiguity raises the buyer's willingness to pay and, in turn, the increased demand allows the seller to earn revenue greater than the buyer's expected valuation under the common prior (i.e., the upper bound on revenue without ambiguity). Of course, if the buyer were to perceive no ambiguity about her purchase decision at the ex ante stage, she would expect to pay more than her valuation and refuse to participate. With ex ante optimism, however, the buyer expects under her most optimistic beliefs to avoid buying when her valuation is low. Thus, ex ante optimism permits to offset the negative effect of interim optimism for the buyer and ensures her participation.

As our second main result, we fully characterize the outcomes in terms of buyer's utility and seller's revenue that can obtain under any ambiguous signal structure. Figure 1 illustrates these outcomes, the left panel for a high α and the right panel for a low α. The yellow triangle indicates the possible outcomes without ambiguity: every combination of utility and revenue summing at most to the prior expected valuation $\mathbb{E}_F[v]$ (the maximal surplus without ambiguity) is attained by some unambiguous signal structure, provided that the revenue is above the lowest possible level q^* (as characterized by Roesler and Szentes, 2017). Ambiguity expands the set of possible outcomes by the blue area. First, ex ante optimism shifts the Pareto frontier (the upper right boundary) to the right. On the frontier without ambiguity, the buyer always buys and revenue equals the price. Ex ante optimism allows for higher utility as the buyer anticipates in the best-case scenario to avoid buying when her valuation is below the price. Clearly, this effect is greater the greater the price, which explains why the frontier with ambiguity is steeper than the frontier without ambiguity. Secondly, interim pessimism can lower revenue down to $(1-\alpha)q^*$, resulting in the lower boundary in Figure 1. Finally, interim optimism together with ex ante optimism can increase revenue beyond $\mathbb{E}_F[v]$, as indicated by the upper extreme point of the blue area. Intuitively, the effect of interim pessimism increases with the degree of ambiguity aversion α, whereas the impact of ex ante optimism and interim optimism decreases with α. This explains the differences between the left and right panel of Figure 1.

The extreme points in Figure 1 that correspond to the buyer-optimal outcome and seller-optimal outcome are of particular interest. These are the outcomes that result when one side of the market can design the information, and we also characterize corresponding optimal ambiguous signal structures. Buyer-optimal signal structures combine interim pessimism with ex ante optimism and exploit these effects as much as possible. Buyer-optimal information could, for example, be designed by the buyer herself (in particular, if she has sufficient market power to affect the seller's pricing, such as governments or corporations who acquire information for procurement projects), by a regulator who regulates product information or provides additional information to raise consumer surplus, or by a sales platform who cares about consumer surplus because of network effects or advertising revenues. Seller-optimal signal structures maximally exploit interim optimism and ex ante optimism. This relates to a seller who has control over what consumers learn and decides on what product information to disclose over which channels. Lastly, note that there are also outcomes under ambiguous information that are Pareto dominated by *every* outcome that is possible without ambiguity, such as the lower left corner of Figure 1. If such information prevails, welfare unambiguously increases when ambiguity is eliminated.

For the welfare interpretation of Figure 1, it is important that we measure consumer surplus with the buyer's ex ante α-max-min expected utility, which accounts for the (ex ante) ambiguity she perceives concerning ex post payoffs. This ex ante utility represents the buyer's preferences over signal structures before observing the signal, and it measures the buyer's willingness to pay to participate. Our general insight that the seller or the buyer can benefit from ambiguity, however, is robust to other measures of consumer surplus: As long as ex ante utility is relevant for the participation decision, the seller can benefit from ex ante and interim optimism. Moreover, interim pessimism can lower the price without exposing the buyer to ambiguity at the ex ante stage. Hence, ambiguity can also increase an ambiguity-neutral measure of consumer surplus.

Our paper contributes to the literature on information design in monopolistic markets and bilateral trade, which so far focused on unambiguous information.[4] When restricted to unambiguous signal structures, our setting corresponds to that of Roesler and Szentes (2017) who characterize all possible combinations of utility and revenue. In similar settings, Ravid et al. (2022) analyze a buyer and a seller who simultaneously choose the signal structure and the price, respectively; Terstiege and Wasser (2020) examine buyer-optimal learning when the seller can add information; and Armstrong and Zhou (2022) consider the case of more than one seller. Bergemann et al. (2015) characterize all possible combinations of utility and revenue when the buyer knows her valuation and the seller receives an unambiguous signal about the buyer's valuation.[5]

Our characterization of buyer-optimal and seller-optimal signal structures relates to papers on endogenously designed ambiguity. Like us, Di Tillio et al. (2017) consider an ambiguity-sensitive buyer who faces an ambiguity-neutral seller. While the buyer's privately known valuation is unambiguous in their case, they allow the seller to design allocation and payment rules that are perceived as ambiguous by the buyer, and they identify environments where doing so increases the seller's revenue. Bose and Renou (2014) characterize the implementable social choice functions in a mechanism-design setting where the designer can add a communication stage to introduce ambiguity into each agent's beliefs about the other agents' types. In both papers, the agents have max-min preferences and the designer exploits the agents' pessimism to deal with incentive constraints. In our setting, by contrast, it is optimal for the buyer to expose herself to ambiguity, as her pessimism induces the seller to lower the price. The seller, on the other hand, can only benefit from the optimism the buyer's α-max-min preferences allow for. Beauchêne et al. (2019) and Kellner and Le Quement (2018) introduce endogenous ambiguity to models of Bayesian persuasion (Kamenica and Gentzkow, 2011) and cheap-talk communication (Crawford and Sobel, 1982), respectively. They consider a sender who can choose to send ambiguous signals to a receiver

[4]See Bergemann and Morris (2019) and Kamenica (2019) for surveys on information design and Bayesian persuasion.

[5]Makris and Renou (2023) and Kartik and Zhong (2026) allow for more general information structures, encompassing both Bergemann et al. (2015) and Roesler and Szentes (2017). In all four papers the set of possible outcomes takes the shape of a triangle. They differ in the lowest possible revenue.

with max-min preferences. Beauchêne et al. (2019) show that ambiguous persuasion typically benefits the sender, whereas Kellner and Le Quement (2018) find that equilibria with ambiguous communication can make both the sender and the receiver better off.

The literature on endogenously designed ambiguity typically assumes that the designer can choose sets of mechanisms or (unambiguous) signal structures and thereby create ambiguity for agents with respect to which element of the set is relevant. The agents then maximize their minimum utility across elements, which corresponds to max-min preferences with a fully ambiguous state that determines the relevant element.[6] In our framework, ambiguous information is designed by conditioning on an exogenously given payoff-irrelevant state the agent has multiple priors about. This provides a transparent and flexible way for formulating constraints on the ambiguity the designer can introduce, and it allows to clarify to what extent results depend on assumptions on the payoff-irrelevant state. For example, many of our results do not require full ambiguity.

Sensitivity to ambiguity is well documented in experimental and empirical studies (see, e.g., Abdellaoui et al., 2011, and Dimmock et al., 2016). In particular, a recent experimental literature (e.g., Kellner et al., 2022; Kops and Pasichnichenko, 2023; Shishkin and Ortoleva, 2023) studies how subjects respond to ambiguous signals in the spirit of those in our setting and in the literature discussed in the two preceding paragraphs. While the results of Kellner et al. (2022) and Kops and Pasichnichenko (2023) are as predicted by models of ambiguity aversion, Shishkin and Ortoleva (2023) find that subjects only react to ambiguous signals if they are beneficial and ignore them if they are harmful. In our setting the ambiguous signals are typically beneficial for the buyer from the ex ante perspective. Hence, all three experiments suggest that the effects emerging from our analysis are relevant.

Related strands of literature study the implications of exogenously given ambiguity for Bayesian persuasion (e.g., Hu and Weng, 2021; Hedlund et al., 2021; Kosterina, 2022; Dworczak and Pavan, 2022) and mechanism design (e.g., Bose et al., 2006; Wolitzky, 2016; Song, 2018; De Castro and Yannelis, 2018). These strands intersect with the literature on robust mechanism design (see Carroll, 2019, for a survey). There, the designer is uncertain about some relevant parameters and seeks for a robust mechanism that is optimal in the worst case, which

[6]Kellner and Le Quement (2018) refer to an exogenously given payoff-irrelevant binary state, and they consider an extension with less than full ambiguity.

is reminiscent of a designer with max-min preferences.[7] In the context of a seller facing one or several privately informed buyers, Bergemann and Schlag (2011), Carrasco et al. (2018), Du (2018), and Brooks and Du (2021) study robust selling mechanisms when the seller is uncertain about the distribution of the buyers' willingness to pay. Carroll (2017) and Che and Zhong (2025) consider the sale of multiple objects to one buyer.

As α-max-min preferences allow incorporating different attitudes towards ambiguity in a tractable way, they have become a popular modeling choice in various applications. For examples, see Chen et al. (2007), Bossaerts et al. (2010), Ahn et al. (2014), Cherbonnier and Gollier (2015), Saghafian (2018), and Beissner et al. (2020). The theoretical interpretation of α-max-min preferences is still an active debate (see, e.g., Frick et al., 2022, and Hartmann, 2023). We contribute to this discussion by showing that α-max-min preferences can imply interim pessimism due to our information design even for ambiguity attitudes α close to 0, that is, close to max-max preferences.

The remainder of this paper is organized as follows. Section 2 discusses an illustrative example. Section 3 presents the model. Section 4 provides the unambiguous benchmark. Section 5 shows that the buyer or the seller can strictly benefit from ambiguity. Section 6 characterizes all possible outcomes. Section 7 analyzes dynamic consistency. Section 8 concludes. The Appendix contains the proofs.

2 An Illustrative Example

Before defining the general model, we illustrate the main effects of ambiguous information in a simple example. Consider a seller who offers an object to a buyer. Initially, both agents are uncertain about the buyer's valuation v, which equals 0 or 1 with probability 1/2 each under their common prior. Before trading, the buyer obtains additional information about v by privately observing a signal from a commonly known signal structure. Then the seller sets the price, and the buyer decides whether to buy.

As a starting point, we consider the following unambiguous signal structure: it sends one of two possible signals, m_1 and m_2, conditional on the buyer's valuation v. If $v = 0$, the signal m_1 occurs with probability 1. If $v = 1$, signals m_1 and m_2 occur with probability 2/3 and 1/3, respectively. Signal m_2, which occurs with unconditional probability 1/6, thus reveals to the buyer that her valuation is $v = 1$.

[7] This interpretation of max-min preferences follows the idea that individuals seek robust decision-making for fear of model misspecification (Hansen and Sargent, 2001).

Upon observing m_1, the buyer updates her prior belief $\mathbb{P}[v = 1] = 1/2$ that her valuation is 1 to the posterior belief $\mathbb{P}[v = 1|m_1] = 2/5$. Accordingly, the buyer's willingness to pay (WTP) is $\mathbb{E}[v|m_1] = 2/5$, which occurs with probability 5/6. The following table summarizes these values:

m	m_1	m_2
$\mathbb{P}[m\|v = 0]$	1	0
$\mathbb{P}[m\|v = 1]$	2/3	1/3
$\mathbb{E}[v\|m]$ = buyer's WTP	2/5	1
seller's prior	5/6	1/6

Given this distribution of WTP, the seller's optimal price is 2/5, which yields expected revenue 2/5. Accordingly, the buyer buys after each signal, and her ex ante expected utility equals the prior expected valuation minus the price, that is, $1/2 - 2/5 = 1/10$.

Let us now introduce ambiguity. To do so, we assume that signal structures can condition not only on the valuation v but also on an independent and payoff-irrelevant state $\omega \in \{\omega_0, \omega_1\}$. The seller believes that $\omega = \omega_0$ with probability π. Yet the buyer perceives ω as ambiguous and finds any probability distribution on $\{\omega_0, \omega_1\}$ plausible, that is, she has multiple priors. To summarize, her set of priors on (v, ω) consists of any distribution such that $\mathbb{P}[(v, \omega)] = \mathbb{P}[\omega]/2$ for all (v, ω) and $\mathbb{P}[\omega_0] \in [0, 1]$. Her preferences are represented by α-max-min expected utility with $\alpha \in (0, 1)$ and full Bayesian updating. This means that the buyer's utility is the sum of α times the expected payoff under the worst-case belief and $(1 - \alpha)$ times the expected payoff under the best-case belief given her set of beliefs (i.e., her priors before observing the signal or her updated priors after observing the signal). We next consider two ambiguous signal structures: one where the buyer faces ambiguity at the interim stage when making the purchase decision and one where the buyer perceives ambiguity at the ex ante stage (i.e., before observing the signal).

The first ambiguous signal structure draws one of the three signals m_1, m_2, and m_3 conditional on valuation v and state ω as follows:

m	m_1	m_2	m_3
$\mathbb{P}[m \mid v = 0, \omega = \omega_1]$	1	0	0
$\mathbb{P}[m \mid v = 1, \omega = \omega_1]$	2/3	1/3	0
$\mathbb{P}[m \mid v = 0, \omega = \omega_0]$	2/3	1/3	0
$\mathbb{P}[m \mid v = 1, \omega = \omega_0]$	0	0	1
$\mathbb{E}[v \mid m]$	best case: 2/5 worst case: 0	best case: 1 worst case: 0	1
buyer's WTP	$(1-\alpha)2/5$	$1-\alpha$	1
seller's prior	$(1-\pi)5/6+\pi/3$	1/6	$\pi/2$

In state ω_1, the signals m_1 and m_2 occur with the same probabilities conditional on v as under the unambiguous signal structure above. In state ω_0, signals m_1 and m_2 occur only if $v = 0$ whereas signal m_3 occurs with probability one if $v = 1$. The signal m_3 thus perfectly reveals to the buyer that her valuation is 1. The meaning of the signals m_1 and m_2, however, is ambiguous to the buyer as it depends on the state ω. Consider signal m_1. In state ω_1, the expected valuation after m_1 is 2/5 as under the unambiguous signal structure. This is higher than in state ω_0 where m_1 reveals the valuation to be 0. Hence, upon observing m_1, the buyer's utility from obtaining the object is $\alpha \cdot 0 + (1-\alpha) \cdot 2/5 = (1-\alpha)2/5$. Similarly, upon observing m_2, the utility is $1-\alpha$. Note that m_1 occurs with probability 5/6 and 1/3 in state ω_1 and ω_0, respectively, and m_2 occurs with probability 1/6 in both states. Therefore, from the seller's perspective, the buyer's WTP is $(1-\alpha)2/5$ with probability $(1-\pi)5/6+\pi/3$, $1-\alpha$ with probability 1/6, and 1 with probability $\pi/2$. Given this distribution of WTP, one can easily verify that the seller's optimal price is $(1-\alpha)2/5$ if the seller's prior probability of ω_0 is sufficiently low, that is, $\pi \leq \min\{7, 12(1-\alpha)\}/15$. Under this condition, ambiguity lowers the buyer's WTP as she is to some extent pessimistic when interpreting the signal, which, in turn, induces the seller to lower the price. We call this effect *interim pessimism* as it occurs at the interim stage when the signal realizes. From the ex ante perspective, the buyer perceives no ambiguity: she anticipates that she will always buy at price $(1-\alpha)2/5$, no matter what signal she observes. Hence, her ex ante utility is $1/2 - (1-\alpha)2/5$, which is greater than under the unambiguous signal structure above.

The second ambiguous signal structure draws m_1, m_2, and m_3 as follows:

m	m_1	m_2	m_3
$\mathbb{P}[m \mid v = 0, \omega = \omega_1]$	1	0	0
$\mathbb{P}[m \mid v = 1, \omega = \omega_1]$	2/3	1/3	0
$\mathbb{P}[m \mid v = 0, \omega = \omega_0]$	0	0	1
$\mathbb{P}[m \mid v = 1, \omega = \omega_0]$	0	1	0
$\mathbb{E}[v \mid m]$ = buyer's WTP	2/5	1	0
seller's prior	$(1-\pi)5/6$	$(1-\pi)/6 + \pi/2$	$\pi/2$

Under this signal structure, the buyer perceives no relevant ambiguity interim when making the purchase decision. Signal m_1 reveals that $\omega = \omega_1$ and implies expected valuation 2/5 as under the unambiguous signal structure, signal m_2 reveals that $v = 1$ independent of the state ω, and signal m_3 reveals that $\omega = \omega_0$ and $v = 0$. From the seller's perspective, m_1 occurs with probability $(1-\pi)5/6$, m_2 with probability $(1-\pi)/6 + \pi/2$, and m_3 with probability $\pi/2$. In the following, we assume $\pi \leq 7/16$. Then, the seller's optimal price is still 2/5 as under the unambiguous signal structure. Now it is from the ex ante perspective that the buyer perceives ambiguity: at price 2/5, she expects payoff 1/10 from always buying in state ω_1 and payoff $(1 - 2/5)/2 = 3/10$ from buying only if her valuation is 1 in state ω_0. Hence, the buyer's ex ante utility is $\alpha/10 + (1-\alpha)3/10$, which is greater than under the unambiguous signal structure above. The reason for this effect is that the buyer is to some extent optimistic about being able to make a better purchase decision upon observing the signal. We call this effect *ex ante optimism.*

We have identified two effects of ambiguous information – interim pessimism and ex ante optimism – that increase the buyer's ex ante utility compared to the unambiguous signal structure considered at the outset. We will show in Section 5 that each of the two effects can increase the buyer's ex ante utility beyond her ex ante utility under the best unambiguous signal structures. Moreover, under the conditions identified in Section 6, combining the two effects maximizes the buyer's ex ante utility among all signal structures.

Finally, we consider a third ambiguous signal structure to illustrate that also the seller can benefit from ambiguity. It draws signals m_1, m_2, and m_3 as follows:

m	m_1	m_2	m_3
$\mathbb{P}[m\|v=0,\omega=\omega_1]$	1	0	0
$\mathbb{P}[m\|v=1,\omega=\omega_1]$	2/3	1/3	0
$\mathbb{P}[m\|v=0,\omega=\omega_0]$	2/5	0	3/5
$\mathbb{P}[m\|v=1,\omega=\omega_0]$	3/5	0	2/5
$\mathbb{E}[v\|m]$	best case: 3/5 worst case: 2/5	1	2/5
buyer's WTP	$(3-\alpha)/5$	1	2/5
seller's prior	$(1-\pi)5/6+\pi/2$	$(1-\pi)/6$	$\pi/2$

Under this signal structure, the buyer perceives no ambiguity after m_2 and m_3. Signal m_2 reveals that $\omega = \omega_1$ and $v = 1$, whereas signal m_3 reveals that $\omega = \omega_0$ and implies expected valuation 2/5. The meaning of signal m_1, however, is ambiguous: the expected valuation after m_1 is 2/5 in state ω_1 (as under the unambiguous signal structure) but 3/5 in state ω_0. Hence, upon observing m_1, the buyer's WTP is $\alpha \cdot 2/5 + (1-\alpha) \cdot 3/5 = (3-\alpha)/5$. Note that the seller is weakly better off than under the unambiguous signal structure above since price 2/5 still yields revenue 2/5. Moreover, given the distribution of WTP under the seller's prior, price $p^* = (3-\alpha)/5$ yields revenue $(1-\pi/2)p^* > 2/5$ if $\pi < 2(1-\alpha)/(3-\alpha)$, that is, if the seller's prior probability of ω_0 is sufficiently low. The seller benefits from a counterpart to interim pessimism that we call *interim optimism*: ambiguity increases the buyer's WTP as she is to some extent optimistic when interpreting signal m_1. Most importantly, if $\alpha < 1/2$ and $\pi < (1-2\alpha)/(3-\alpha)$, interim optimism permits the seller to earn even revenue $(1-\pi/2)p^* > 1/2$. By contrast, under any unambiguous signal structure revenue can at most equal the prior expected valuation 1/2. Revenue above 1/2 raises the question whether the buyer is willing to participate. From the ex ante perspective, the buyer anticipates that she will buy at price p^* after signals m_1 and m_2 but not after signal m_3. She expects payoff $1/2 - p^*$ from always buying in state ω_1 and payoff $(3/5 - p^*)/2$ from buying only after signal m_1 (where the expected valuation is $3/5 > p^*$) in state ω_0. Accordingly, her ex ante utility is $\alpha(1/2 - p^*) + (1-\alpha)(3/5 - p^*)/2 = \alpha^2/10$. Hence, ex ante optimism about avoiding to buy when her expected valuation is low in state ω_0 guarantees the buyer a positive ex ante utility from participation even when the seller earns revenue above 1/2. In Section 5 we find that interim and ex ante optimism may increase revenue for any $\alpha < 1$.

3 The Model

Consider a buyer (she) and a seller (he). They are uncertain about the buyer's valuation v of the object offered by the seller. Before trading, the buyer obtains additional information about her valuation through observing a private signal from a commonly known signal structure. Then the seller makes a take-it-or-leave-it price offer p. Lastly, the buyer decides whether to accept the seller's offer. If the buyer accepts, the buyer's payoff is $v - p$ and the seller's payoff is p. Otherwise, each agent's payoff is zero. Payoffs are also zero if the buyer rejects trading before observing the signal.

We use an Anscombe and Aumann (1963) framework with α-max-min expected utility. The state of the world (v, ω) has two components: the buyer's valuation $v \in [0, u]$ with $u > 0$ and a payoff-irrelevant state $\omega \in \Omega$ with $\Omega \subset \mathbb{R}$.[8] Both agents agree that v and ω are independent. Hence, any prior about the state of the world can be represented by a cumulative distribution function (CDF) $P(v, \omega) = F(v)G(\omega)$, where F is a CDF on $[0, u]$ representing a prior about valuation v and G is a CDF on Ω representing a prior about state ω. Buyer and seller both have the same unique prior F about the valuation v, which satisfies $0 < F(v) < 1$ for all $v \in (0, u)$.[9] They differ, however, in their perception of Ω.

The buyer is ambiguity sensitive. Her preferences can be represented by a parameter $\alpha \in [0, 1]$ and a convex and weakly compact set of priors $\mathcal{G}$ on Ω such that the buyer's utility from any act a (a mapping from states of the world to lotteries over payoffs) is[10]

$$\alpha \min_{G \in \mathcal{G}} \mathbb{E}_{F,G}[a] + (1 - \alpha) \max_{G \in \mathcal{G}} \mathbb{E}_{F,G}[a].$$

The seller is ambiguity neutral: he has a unique prior G_S on Ω, and his expected utility from an act a is simply $\mathbb{E}_{F,G_S}[a]$.[11] In the spirit of a common-prior assumption, we assume $G_S \in \mathcal{G}$. Lastly, we assume that the buyer uses full Bayesian updating (Pires, 2002).

[8]Public randomization devices, which are common, e.g., in game theory, also use such an abstract state space Ω that is not (directly) relevant for payoffs.

[9]That is, the support of the prior F is a subset of $[0, u]$ that contains at least 0 and u. Recall that the support of a distribution is the smallest closed set of values that has probability one.

[10]Having multiple priors over ω, the buyer has multiple priors P over the full state (v, ω). Our formulation of the utility is equivalent to $\alpha \min_P \mathbb{E}_P[a] + (1 - \alpha) \max_P \mathbb{E}_P[a]$.

[11]Being ambiguity neutral, the seller deals with the ambiguity differently than the ambiguity-sensitive buyer. In the framework of α-max-min preferences, the seller's set of distributions is a singleton. Di Tillio et al. (2017) combine an ambiguity-averse agent with an ambiguity-neutral principal in a similar way.

Before making her purchase decision, the buyer learns additional information from a signal structure that sends a stochastic signal conditional on the state of the world. Specifically, a *signal structure* H consists of a CDF $H_{v,\omega}$ on M for each state (v,ω), where $M \subset \mathbb{R}_+$ is a compact set and $H_{v,\omega}$ is measurable with respect to v and ω. We let $\bar{H}_\omega = \int_0^u H_{v,\omega}\mathrm{d}F(v)$ denote the distribution of the signal conditional only on ω. A signal structure is *unambiguous* if $H_{v,\omega}$ is constant in ω.

We now turn to the buyer's purchase decision. Suppose the seller offers the object at price $p > 0$ and the buyer has observed signal $m \in M$ from the signal structure H. The buyer buys if, given her set of posteriors, the utility from buying, $v - p$, is greater than or equal to zero (i.e., her reservation utility). If the buyer were to know ω, she would simply compare the price and $\mathbb{E}[v|m,\omega]$, that is, the expected valuation given signal m and state ω.[12] As the buyer does not know ω, she updates each prior $G \in \mathcal{G}$ about the payoff-irrelevant state ω to a posterior $\hat{G}^G_m$, resulting in the set of posteriors $\{\hat{G}^G_m : G \in \mathcal{G}\}$.[13] For example, the prior G is updated to the posterior

$$\hat{G}^G_m(\omega) = \frac{\int_{\omega' \leq \omega} \bar{h}_{\omega'}(m)\mathrm{d}G(\omega')}{\int_\Omega \bar{h}_{\omega'}(m)\mathrm{d}G(\omega')} \tag{1}$$

if $\int_\Omega \bar{h}_{\omega'}(m)\mathrm{d}G(\omega') > 0$ and the distribution $\bar{H}_\omega$ admits a density $\bar{h}_\omega(m)$ for each $\omega \in \Omega$. Then, the buyer's utility from buying is[14]

$$\alpha \min_{G\in\mathcal{G}} \int_\Omega (\mathbb{E}[v|m,\omega] - p)\mathrm{d}\hat{G}^G_m(\omega) + (1-\alpha) \max_{G\in\mathcal{G}} \int_\Omega (\mathbb{E}[v|m,\omega] - p)\mathrm{d}\hat{G}^G_m(\omega).$$

Hence, upon observing signal m, the buyer's *willingness to pay* is

$$\hat{v}(m) = \alpha \min_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) + (1-\alpha) \max_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega), \tag{2}$$

that is, she buys if and only if $\hat{v}(m) \geq p$.

[12] Formally, the conditional expectation $\mathbb{E}[v|m,\omega]$ is characterized by the condition that for all $Y \in \mathcal{B}(M)$, $\int_Y \mathbb{E}[v|m,\omega]\mathrm{d}\bar{H}_\omega(m) = \int_0^u \int_Y v\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)$ for a given $\omega \in \Omega$, with $\mathcal{B}(\cdot)$ denoting the respective Borel σ-algebra. For unambiguous signal structures, $\mathbb{E}[v|m,\omega]$ has to be constant in ω.

[13] $\hat{G}^G_m$ is the conditional distribution of the state ω given the prior G and that signal m was drawn from $\bar{H}_\omega$. Formally, $\hat{G}^G_m$ is characterized by $\int_\Omega \int_Y \int_Z \mathrm{d}\hat{G}^G_m(\omega)\mathrm{d}\bar{H}_{\omega'}(m)\mathrm{d}G(\omega') = \int_Z \int_Y \mathrm{d}\bar{H}_\omega(m)\mathrm{d}G(\omega)$ for all $Y \in \mathcal{B}(M)$ and all $Z \in \mathcal{B}(\Omega)$. If m is not in the support of $\bar{H}_{\tilde{\omega}}$ for some $\tilde{\omega} \in \Omega$, we set $\mathbb{P}_{\hat{G}^G_m}[\tilde{\omega}] = \int_{\{\tilde{\omega}\}} \mathrm{d}\hat{G}^G_m(\omega) = 0$ for all $G \in \mathcal{G}$.

[14] Notice that $\int_\Omega (\mathbb{E}[v|m,\omega] - p)\mathrm{d}\hat{G}^G_m(\omega)$ is equivalent to $\int_{[0,u]\times\Omega} (v-p)\mathrm{d}\hat{P}^P_m(v,\omega)$ where $\hat{P}^P_m$ is the posterior belief after signal m for a prior $P = F \cdot G$ about the full state (v,ω).

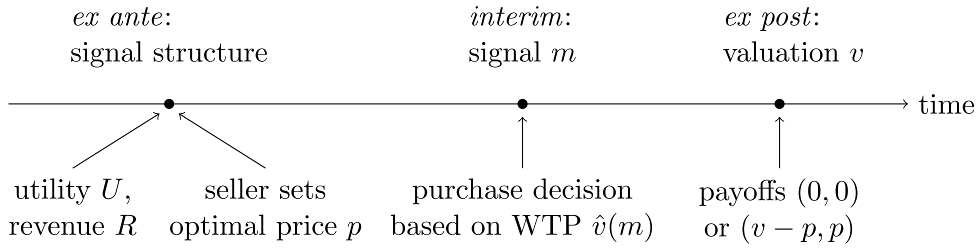

Figure 2: Timing

When setting the price, the seller knows the signal structure H but not the signal m observed by the buyer. Given the seller's unique prior G_S about the payoff-irrelevant state ω and how the buyer will make her purchase decision, the seller chooses price p to maximize his expected *revenue*

$$p \int_{\Omega} \int_{M} \mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}_{\omega}(m) \mathrm{d}G_S(\omega), \tag{3}$$

where $\mathbf{1}(\cdot)$ is the indicator function.

Finally, we consider the buyer's preferences over signal structures from the ex ante perspective, that is, before she learns the signal. We use sophistication in the sense of Siniscalchi (2011): the buyer anticipates any preference reversals.[15] A signal structure H and a price p correspond to the act that maps state (v, ω) to the utility $(v - p)\mathbf{1}(\hat{v}(m) \geq p)$ where signal m is drawn from the distribution $H_{v,\omega}$. The buyer's *ex ante utility* is the utility of this act given her set of priors about (v, ω). A signal structure H *implements* price p, revenue R, and ex ante utility U if p maximizes (3), the maximum of (3) is R, and

$$\begin{aligned} U = \quad & \alpha \min_{G \in \mathcal{G}} \int_{\Omega} \int_0^u \int_M (v - p)\mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) \mathrm{d}G(\omega) \\ & + (1 - \alpha) \max_{G \in \mathcal{G}} \int_{\Omega} \int_0^u \int_M (v - p)\mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) \mathrm{d}G(\omega) \geq 0. \end{aligned}$$

A price p, revenue R, and ex ante utility U are *implementable* if there is a signal structure that implements them. Figure 2 summarizes the timing.

[15]See Section 7 for more details.

4 Benchmark: Unambiguous Signal Structures

As a benchmark, we first focus on unambiguous signal structures H. For such H, there is a CDF H_v for each v such that $H_{v,\omega} = H_v$ for all payoff-irrelevant states ω. Thus, we will identify unambiguous signal structures H with their distributions H_v, and we let $\bar{H} = \int_0^u H_v \mathrm{d}F(v)$. We next review the analysis of unambiguous signal structures by Roesler and Szentes (2017).

For $0 \leq q \leq B \leq u$, define the two-parameter family of CDFs J_q^B on $[q, B]$ as

$$J_q^B(m) = \begin{cases} 1 - \frac{q}{m} & \text{if } m \in [q, B), \\ 1 & \text{if } m = B, \end{cases}$$

which correspond to censored Pareto distributions.[16] Consider an unambiguous signal structure H with $M = [q, B]$ and H_v such that $\bar{H} = J_q^B$ and $\mathbb{E}[v|m] = m$. Having observed signal m and facing price p, the buyer buys if and only if $\hat{v}(m) = m \geq p$, so that the seller's revenue at any price $p \in [q, B]$ is

$$p \int_q^B \mathbf{1}(m \geq p) \mathrm{d}J_q^B(m) = q$$

whereas any other price yields less revenue. That is, the seller faces unit-elastic demand, rendering him indifferent between all prices between q and B. Hence, this signal structure implements price q and ex ante utility $\mathbb{E}_F[v] - q$ because the buyer always buys given price q.

As is well known (see, e.g., Kolotilin, 2018), given $\bar{H}$ and $\mathbb{E}[v|m] = m$, a corresponding signal structure H with distributions H_v exists if and only if the prior F on v is a mean-preserving spread of $\bar{H}$. Thus, the lowest price that can be implemented with a signal structure as defined in the preceding paragraph is

$$q^* = \min_q \left\{ q \in [0, u] : F \text{ is a mean-preserving spread of } J_q^B \text{ for some } B \right\}.$$

Let B^* be such that F is a mean-preserving spread of $J_{q^*}^{B^*}$. Roesler and Szentes (2017) show that q^* is the lowest revenue and $\mathbb{E}_F[v] - q^*$ the highest ex ante utility implementable by unambiguous signal structures. Moreover, every outcome (U, R) of revenue $R \geq q^*$ and ex ante utility $U \geq 0$ such that $R + U \leq \mathbb{E}_F[v]$ is implementable (see the yellow area in Figure 1 for illustration).

[16]For $q = 0$, we define $J_0^B(0) = 1$.

Proposition 1 (Roesler and Szentes, 2017)**.**

(i) There is an unambiguous signal structure H^* with distributions H_v^* such that $\bar{H}^* = \int_0^u H_v^* \mathrm{d}F(v) = J_{q^*}^{B^*}$ and $\mathbb{E}[v|m] = m$ that implements ex ante utility $\mathbb{E}_F[v] - q^*$. No unambiguous signal structure implements a larger ex ante utility.

(ii) An outcome (U, R) is implementable by unambiguous signal structures if and only if $R \geq q^*$ and $0 \leq U \leq \mathbb{E}_F[v] - R$.

Unlike the buyer-optimal signal structures H^* in Proposition 1(i), seller-optimal unambiguous signal structures are straightforward: any H that provides no information about v (i.e., H_v is constant in v) permits the seller to extract the entire surplus by charging price $\mathbb{E}_F[v]$. Accordingly, without ambiguity the highest implementable revenue is $R = \mathbb{E}_F[v]$, as implied by Proposition 1(ii).

Proposition 1 also applies to ambiguous signal structures if the buyer is ambiguity neutral like the seller, that is, $\mathcal{G} = \{G_S\}$. Then, the set of implementable revenue and ex ante utility remains unchanged. If the buyer is ambiguity sensitive, however, there are ambiguous signal structures that implement ex ante utility above the upper bound identified in Proposition 1(i) and other ambiguous signal structures that implement revenue above the upper bound $\mathbb{E}_F[v]$, as we show in the next section. In Section 6, we further show how ambiguity expands the set of outcomes identified in Proposition 1(ii).

5 Benefits of Ambiguous Information

Can ambiguity be beneficial for the ambiguity-sensitive buyer or the ambiguity-neutral seller? The next theorem provides an unambiguous answer: there are ambiguous signal structures that the buyer strictly prefers and ambiguous signal structures that the seller strictly prefers to any unambiguous signal structure. We state the theorem under the following assumption, part of which ensures that the buyer is not ambiguity neutral.

Assumption A1. There is a payoff-irrelevant state $\omega_0 \in \Omega$ such that $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = \mathbb{P}_{G_S}[\omega_0] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = \delta > 0$.

Under Assumption A1, there is a state ω_0 corresponding to an event (black swan, pandemic, financial crisis) that the seller deems entirely unlikely, while the buyer faces ambiguity concerning the event.[17] The value δ measures the amount of ambiguity concerning ω_0 and $\delta > 0$ ensures that the buyer perceives at least a minimal amount of ambiguity.[18]

Theorem 1. *Assume A1 and* $\alpha \in [0, 1]$*.*

(i) With ambiguous signal structures greater ex ante utility for the buyer is implementable than with unambiguous signal structures.

(ii) With ambiguous signal structures greater revenue for the seller is implementable than with unambiguous signal structures if $\alpha \neq 1$*.*

To prove the theorem and illustrate the underlying effects, we construct in the following three subsections ambiguous signal structures that implement superior outcomes. The buyer benefits from two separate effects: interim pessimism and ex ante optimism. Interim pessimism reduces the buyer's willingness to pay after learning the signal. Reacting to this decreased demand, the seller lowers the price, which increases the buyer's payoffs and ex ante utility whenever ambiguity attitude $\alpha > 0$ makes her consider pessimistic cases. Ex ante optimism allows the buyer in some states to buy only if her valuation is high without inducing the seller to increase the price. This increases the buyer's ex ante utility whenever ambiguity attitude $\alpha < 1$ makes her consider optimistic cases. Subsections 5.1 and 5.2 introduce the two effects in detail, presenting signal structures that implement the outcomes IP and AO illustrated in Figure 3. Theorem 1(i) immediately follows from Propositions 2 and 3 established there.

The seller benefits from a combination of interim optimism and ex ante optimism whenever $\alpha < 1$. Interim optimism increases the buyer's willingness to pay after learning the signal, such that the increased demand allows the seller to earn revenue greater than $\mathbb{E}_F[v]$. At the same time, ex ante optimism ensures that the buyer's participation constraint is satisfied despite the high revenue: the buyer is optimistic about avoiding to pay the high price when her valuation is low in some states that the seller deems unlikely. Subsection 5.3 presents a corresponding signal structure, which implements outcome SE in Figure 3, and establishes Proposition 4 to prove Theorem 1(ii).

[17] A state ω_0 such that $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = \mathbb{P}_{G_S}[\omega_0] = 0$ exists if Ω has infinitely many elements. In this case, Assumption A1 is satisfied by ϵ-contamination as in Auster (2018), Song (2018) and Bose et al. (2006) or δ-ambiguity as in Kocherlakota and Song (2019) and Song (2023). We further discuss the assumption that $G_S[\omega_0] = 0$ at the end of Subsection 5.3.

[18] The literature on endogenously designed ambiguity (e.g., Beauchêne et al., 2019) typically relies on a sufficiently rich set Ω of fully ambiguous states, which corresponds to $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_G[\omega] = \delta = 1$ for all $\omega \in \Omega$.

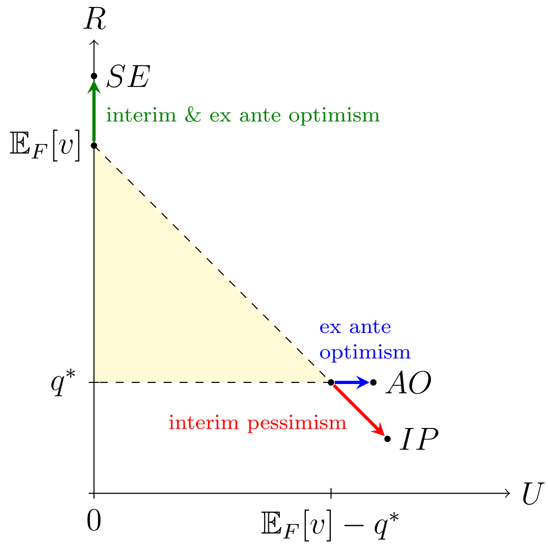

Figure 3: The outcomes in the yellow triangle are implementable without ambiguity (cf. Section 4). The outcomes IP, AO and SE are implemented by the corresponding signal structures in Subsections 5.1, 5.2, and 5.3, exploiting the effects indicated by the colored arrows.

5.1 Benefit for the Buyer I: Interim Pessimism

To study interim pessimism, consider the signal structure $IP = H$ with signals $M = [q^*, B^*] \cup \{m_+\}$ where $m_+ > B^*$ and with distributions[19]

$$H_{v,\omega}(m) = \begin{cases} J_{q^*}^{B^*}(m) & \text{if } \omega = \omega_0 \text{ and } v = 0, \\ \mathbf{1}(m \geq m_+) & \text{if } \omega = \omega_0 \text{ and } v > 0, \\ H_v^*(m) & \text{if } \omega \neq \omega_0. \end{cases}$$

That is, in state ω_0, a signal is randomly drawn from the interval $[q^*, B^*]$ if the valuation is zero and the signal m_+ occurs if the valuation is positive. Accordingly, signal m_+ yields an expected valuation of $\mathbb{E}[v|m_+, \omega_0] = \mathbb{E}_F[v|v > 0]$ and all signals $m \in [q^*, B^*]$ yield $\mathbb{E}[v|m, \omega_0] = 0$. In all states $\omega \neq \omega_0$, IP coincides with the unambiguous benchmark H^*, so that $\bar{H}_\omega = J_{q^*}^{B^*}$ and $\mathbb{E}[v|m, \omega] = m$ for all $m \in [q^*, B^*]$.

For simplicity, we temporarily assume $F(0) > 0$ and Assumption A1 with amount of ambiguity $\delta = 1$. That is, the buyer's set of priors $\mathcal{G}$ contains a distribution that assigns probability zero *and* a distribution that assigns probability one to the state ω_0. Under signal structure IP, every signal $m \in [q^*, B^*]$ implies

[19]Recall that Proposition 1(i) characterizes an optimal unambiguous signal structure H^*.

an expected valuation of zero in the worst case and of m in the best case. Hence, signal structure IP induces willingness to pay $\hat{v}(m) = (1-\alpha)m + \alpha 0 = (1-\alpha)m$ for all signals $m \in [q^*, B^*]$. Recall that the seller assigns probability zero to the state ω_0 under his prior G_S. From his perspective, signal m is drawn from $J_{q^*}^{B^*}$, so that he faces unit-elastic demand of $(1-\alpha)q^*/p$ for prices $(1-\alpha)q^* \leq p \leq (1-\alpha)B^*$. Thus, the construction scales down the entire demand by a factor $1-\alpha$. Therefore, the price $p = (1-\alpha)q^*$ is optimal for the seller. Ex ante, the buyer anticipates that she is going to buy the object for that price after all signals and, hence, in all states of the world. Consequently, her ex ante utility is $\mathbb{E}_F[v] - (1-\alpha)q^*$, which is greater than the benchmark if $\alpha > 0$.

The following proposition generalizes our insights, allowing for $F(0) = 0$ and arbitrary δ. In its proof we construct a family of signal structures $IP(\gamma)$ indexed by $\gamma \geq 0$, which includes the signal structure $IP = IP(0)$ as a special case.

Proposition 2. *Assume $\alpha > 0$ and A1. For each $\gamma \in [0, \mathbb{E}_F[v]/B^*]$ such that $F(\gamma q^*) > 0$, there exists a signal structure $IP(\gamma)$ that implements ex ante utility*

$$U = \mathbb{E}_F[v] - \left(1 - \alpha(1-\gamma)\frac{\delta F(\gamma q^*)}{\delta F(\gamma q^*) + 1 - \delta}\right) q^* > \mathbb{E}_F[v] - q^*$$

and revenue $\mathbb{E}_F[v] - U$. That is, interim pessimism via signal structure $IP(\gamma)$ increases the buyer's ex ante utility beyond the unambiguous benchmark in Proposition 1(i).

Consider the special case where the buyer has max-min preferences (Gilboa and Schmeidler, 1989), which corresponds to $\alpha = 1$. If there is also full ambiguity ($\delta = 1$), interim pessimism becomes such a strong effect that it allows to push the price down to zero and, thus, to let the buyer extract all the surplus. For $\alpha = 1$, the sum of ex ante utility and revenue cannot exceed $\mathbb{E}_F[v]$ under any signal structure, as the buyer is pessimistic regarding any ambiguity she perceives at the ex ante stage. For $\alpha < 1$, this is no longer true. Ambiguity at the ex ante stage can then benefit the buyer, as we show next.

5.2 Benefit for the Buyer II: Ex Ante Optimism

To study ex ante optimism, consider the signal structure $AO = H$ with signals $M = [q^*, B^*] \cup \{m_l, m_h\}$ where $B^* < m_l < m_h$ and with distributions

$$H_{v,\omega}(m) = \begin{cases} \mathbf{1}(m \geq m_l) & \text{if } \omega = \omega_0 \text{ and } v \leq q^*, \\ \mathbf{1}(m \geq m_h) & \text{if } \omega = \omega_0 \text{ and } v > q^*, \\ H_v^*(m) & \text{if } \omega \neq \omega_0. \end{cases}$$

That is, in state ω_0, signal m_l occurs if $v \leq q^*$ but signal m_h occurs if $v > q^*$. Then signal m implies an expected valuation of

$$\mathbb{E}[v|m, \omega_0] = \begin{cases} \mathbb{E}_F[v|v \leq q^*] & \text{if } m = m_l, \\ \mathbb{E}_F[v|v > q^*] & \text{if } m = m_h. \end{cases}$$

In all states $\omega \neq \omega_0$, the signal structure AO coincides with the unambiguous benchmark H^*, so that $\bar{H}_\omega = J_{q^*}^{B^*}$ and $\mathbb{E}[v|m, \omega] = m$ for all $m \in [q^*, B^*]$. Interim, there is no ambiguity for the buyer, as she can infer from her signal whether the state ω_0 has occurred. Therefore, her willingness to pay is

$$\hat{v}(m) = \begin{cases} \mathbb{E}[v|m, \omega_0] & \text{if } m \in \{m_l, m_h\}, \\ m & \text{otherwise.} \end{cases}$$

Assume A1. Then, from the point of view of the seller, the signals m_l and m_h occur with probability zero, so that she faces unit-elastic demand for prices $p \in [q^*, B^*]$. Therefore, price $p = q^*$ is optimal for the seller. At this price, the buyer buys after all signals $m \neq m_l$ but does not buy after signal m_l. Ex ante, the buyer anticipates that, in the best case (ω_0), she buys if and only if her valuation exceeds the price and that, in the worst case (any $\omega \neq \omega_0$), she always buys. For simplicity, we temporarily assume that the amount of ambiguity is $\delta = 1$. Then, the buyer's ex ante utility is

$$\alpha \mathbb{E}_F[v - q^*] + (1 - \alpha)\mathbb{E}_F[\max\{0, v - q^*\}],$$

which is greater than the unambiguous benchmark if $\alpha < 1$. The seller's revenue is q^* as in the unambiguous benchmark. Whereas the seller correctly anticipates that the buyer will not buy after signal m_l, this signal occurs with probability zero under his prior G_S.[20] The following proposition generalizes our insights to arbitrary δ.

Proposition 3. *Assume $\alpha < 1$ and A1. Signal structure AO implements ex ante utility*

$$\tilde{\alpha}\mathbb{E}_F[v - q^*] + (1 - \tilde{\alpha})\mathbb{E}_F[\max\{0, v - q^*\}] > \mathbb{E}_F[v - q^*]$$

with $\tilde{\alpha} = \alpha + (1 - \alpha)(1 - \delta)$ and revenue q^*. That is, ex ante optimism via signal structure AO increases the buyer's ex ante utility beyond the unambiguous benchmark in Proposition 1(i).

Signal structures IP and AO benefit the buyer via different effects. In Section 6 below, we find that the highest implementable ex ante utility is attained by combining IP and AO (see Corollary 1).

5.3 Benefit for the Seller

Assume A1 and $\alpha < 1$. To show that the seller can benefit from ambiguity, we use the signal structure $SE = H$ with signals $M = \{m_1, m_2, m_3\}$ that is defined as follows. Let $\epsilon > 0$ be such that the prior F is a mean-preserving spread of the distribution that draws $\mathbb{E}_F[v] - \epsilon$ and $\mathbb{E}_F[v] + \epsilon$ each with probability 1/2.[21] Define $\sigma = (1 - \alpha)\delta/(2 - (1 - \alpha)\delta) \in (0, 1]$. In state ω_0, H draws m_1 and m_3 such that

$$\mathbb{P}_{\bar{H}_{\omega_0}}[m] = \begin{cases} 1/2 & \text{if } m = m_1, \\ 1/2 & \text{if } m = m_3, \end{cases} \quad \text{and} \quad \mathbb{E}[v|m, \omega_0] = \begin{cases} \mathbb{E}_F[v] - \epsilon & \text{if } m = m_1, \\ \mathbb{E}_F[v] + \epsilon & \text{if } m = m_3, \end{cases}$$

whereas in any state $\omega \neq \omega_0$, H draws m_2 and m_3 such that

$$\mathbb{P}_{\bar{H}_{\omega}}[m] = \begin{cases} 1/2 & \text{if } m = m_2, \\ 1/2 & \text{if } m = m_3, \end{cases} \quad \text{and} \quad \mathbb{E}[v|m, \omega] = \begin{cases} \mathbb{E}_F[v] + \sigma\epsilon & \text{if } m = m_2, \\ \mathbb{E}_F[v] - \sigma\epsilon & \text{if } m = m_3. \end{cases}$$

Note that the definition of ϵ and $\sigma \leq 1$ ensures that H is consistent with F.

For simplicity, we temporarily assume $\delta = 1$. Then, $\sigma = (1 - \alpha)/(1 + \alpha)$. As signal m_3 implies an expected valuation of $\mathbb{E}_F[v] - \sigma\epsilon$ in the worst case and of $\mathbb{E}_F[v] + \epsilon$ in the best case, the buyer's willingness to pay is $\hat{v}(m_3) = \mathbb{E}_F[v] - \alpha\sigma\epsilon + (1 - \alpha)\epsilon = \mathbb{E}_F[v] + \sigma\epsilon$. From the other signals the buyer infers whether ω_0 occurred, so that $\hat{v}(m_1) = \mathbb{E}_F[v] - \epsilon$ and $\hat{v}(m_2) = \mathbb{E}_F[v] + \sigma\epsilon$. As the willingness to

[20]Notice that seller and buyer expect each signal $m \in [q^*, B^*)$ with probability zero, too.

[21]Such an ϵ exists because we assumed $0 < F(v) < 1$ for all $v \in (0, u)$, i.e., F is not degenerate.

pay is $\hat{v}(m) = \mathbb{E}_F[v] + \sigma\epsilon$ with probability one under his prior G_S, the seller sets price $p = \mathbb{E}_F[v] + \sigma\epsilon$ and earns revenue $p > \mathbb{E}_F[v]$, which is greater than under any unambiguous signal structure. The buyer anticipates ex ante that, in the worst case (any $\omega \neq \omega_0$), she always buys and, in the best case (ω_0), she does not buy when her expected valuation is $\mathbb{E}_F[v] - \epsilon < p$ after signal m_1. Consequently, her ex ante utility is

$$\alpha(\mathbb{E}_F[v] - p) + (1 - \alpha)\frac{1}{2}(\mathbb{E}_F[v] + \epsilon - p) = 0.$$

To summarize, ex ante optimism about her purchase decision offsets the negative effect for the buyer of interim optimism (caused by ambiguity after signal m_3), which permits revenue $p > \mathbb{E}_F[v]$ for the seller. The following proposition generalizes our insights to arbitrary δ.

Proposition 4. *Assume $\alpha < 1$ and A1. Signal structure SE implements ex ante utility* 0 *and revenue*

$$\mathbb{E}_F[v] + \sigma\epsilon > \mathbb{E}_F[v].$$

That is, interim optimism and ex ante optimism via signal structure SE increase the seller's revenue beyond the unambiguous benchmark in Proposition 1(ii).

Whereas Assumption A1 requires that the seller places probability $\mathbb{P}_{G_S}[\omega_0] = 0$ on some payoff-irrelevant state ω_0, this is not necessary for SE to be beneficial. If $\mathbb{P}_{G_S}[\omega_0] > 0 = \min_{G \in \mathcal{G}} \mathbb{P}[\omega_0]$, price $p = \mathbb{E}_F[v] + \sigma\epsilon$ yields revenue $(1 - \mathbb{P}_{G_S}[\omega_0]/2)p$. This is greater than $\mathbb{E}_F[v]$ if $\mathbb{P}_{G_S}[\omega_0] < 2\sigma\epsilon/(\mathbb{E}_F[v] + \sigma\epsilon)$. That is, Theorem 1(ii) continues to hold as long as $\mathbb{P}_{G_S}[\omega_0]$ is sufficiently small. Moreover, also the example in Section 2 does not impose $\mathbb{P}_{G_S}[\omega_0] = 0$ when illustrating interim pessimism, ex ante optimism, and interim optimism.[22]

6 The Value of Ambiguous Information

In this section, we fully characterize all implementable outcomes in terms of buyer's utility and seller's revenue. Our analysis proceeds as follows: We first derive bounds on implementable outcomes in Subsection 6.1. In Subsection 6.2, we then show that any outcome within these bounds can indeed be attained. Lastly, we discuss buyer-optimal and seller-optimal signal structures in Subsection 6.3.

[22] For the tractable case where F has support $\{0, u\}$, $\min_{G \in \mathcal{G}} \mathbb{P}[\omega_0] = 0$, and $\max_{G \in \mathcal{G}} \mathbb{P}[\omega_0] = 1$, one can further show that modified versions of IP and AO implement ex ante utility $U > \mathbb{E}_F[v] - q^*$ if $\mathbb{P}_{G_S}[\omega_0] > 0$ is sufficiently small, so that also Theorem 1(i) continues to hold.

6.1 Bounds on Implementable Outcomes

We will establish bounds on implementable outcomes in Proposition 5 below using two lemmas. Clearly, the buyer's ex ante utility is decreasing in the price. Thus, an upper bound on the ex ante utility corresponds to a lower bound on implementable revenue and price. To obtain this lower bound in Lemma 2, we partially characterize the distributions of the buyer's willingness to pay the seller may face in Lemma 1. From the seller's point of view, signal structure H induces the *distribution of* (the buyer's) *willingness to pay*

$$K(v) = \int_{\Omega} \int_{M} \mathbf{1}(\hat{v}(m) \leq v) \mathrm{d}\bar{H}_{\omega}(m) \mathrm{d}G_S(\omega). \tag{4}$$

Accordingly, a signal structure that induces distribution K implements revenue $R = \max_p \int_p^u p \mathrm{d}K(v)$. We say a distribution of willingness to pay K is *feasible* if there is a signal structure that induces K. To characterize the set of feasible distributions, let $\mathcal{K}$ denote the set of all distributions K such that the support of K is a subset of $[0, u]$ and

$$\int_0^u \min\{v, (1-\alpha)u\} \mathrm{d}K(v) \geq (1-\alpha)\mathbb{E}_F[v].$$

Lemma 1. *A distribution of willingness to pay K is feasible only if $K \in \mathcal{K}$.*

Lemma 1 proves that $\mathcal{K}$ is a superset of the set of feasible distributions of willingness to pay. All distributions in $\mathcal{K}$ satisfy the intuitive condition $\int_0^u v \mathrm{d}K(v) \geq (1-\alpha)\mathbb{E}_F[v]$. That is, the expected willingness to pay is at least the expected valuation scaled down by the factor $1-\alpha$, which corresponds to the largest possible reduction in demand through interim pessimism as discussed in Subsection 5.1.

We now consider the problem of choosing a distribution $K \in \mathcal{K}$ to minimize the seller's revenue $\max_p \int_p^u p \mathrm{d}K(v)$. For that purpose, the value q_0 such that the mean of the distribution $J_{q_0}^u$ is equal to the mean of the prior F will play an important role. Formally, we define $q_0 \in (0, \mathbb{E}_F[v])$ as the unique solution to

$$\mathbb{E}_{J_{q_0}^u}[v] = \mathbb{E}_F[v], \qquad \text{where } \mathbb{E}_{J_{q_0}^u}[v] = q_0(\ln(u) - \ln(q_0)) + q_0.$$

Whereas $J_{q_0}^u \in \mathcal{K}$ yields revenue q_0, the following lemma shows that no distribution of willingness to pay $K \in \mathcal{K}$ yields less revenue than $(1-\alpha)q_0$, that is, the revenue from scaling down $J_{q_0}^u$ by the factor $1-\alpha$.

Lemma 2. *Every distribution $K \in \mathcal{K}$ yields at least revenue $(1-\alpha)q_0$.*

As every feasible distribution of willingness to pay is in the set $\mathcal{K}$ by Lemma 1, Lemma 2 implies that any signal structure implements at least revenue $(1-\alpha)q_0$, which establishes part (i) of the following proposition. As the price is always at least as high as the revenue, also no lower price than $(1-\alpha)q_0$ is implementable. Part (ii) of the proposition establishes a general upper bound on the ex ante utility for any given revenue.

Proposition 5.

(i) No signal structure implements less revenue than $(1-\alpha)q_0$.

(ii) Every signal structure yielding revenue R implements ex ante utility

$$U \leq \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\mathbb{E}_F[\max\{v - R, 0\}].$$

Intuitively, for any given price p, the buyer obtains the highest possible payoff when buying if and only if her valuation v is greater than p. Hence, in the best case, the buyer's ex ante expected payoff is at most $\mathbb{E}_F[\max\{v-p, 0\}]$. The revenue cannot be larger than the price p. Thus, we get the term weighted with $1-\alpha$ in Proposition 5(ii). The term weighted with α corresponds to an upper bound on the ex ante utility in the worst case: On the one hand, the expected utility from obtaining the object is at most $\mathbb{E}_F[v]$. On the other hand, the expected payment under the worst-case prior in $\mathcal{G}$ is at least equal to the expected payment under the prior G_S, which corresponds to the seller's revenue R.

Notice that Proposition 5 does not require Assumption A1. Every implementable outcome (U, R) satisfies the general necessary conditions provided by parts (i) and (ii). We next identify assumptions under which these conditions are also sufficient.

6.2 All Implementable Outcomes

We will now establish that for any outcome within the bounds given in Proposition 5 there is a signal structure that implements this outcome. We need two assumptions. Assumption A2 restricts the prior.

Assumption A2. The prior F has a mass point at zero, that is, $F(0) > 0$. Moreover, there is a $q \in (0, u)$ so that the prior F is a mean-preserving spread of J_q^u, that is,

$$\int_0^u F(v)\mathrm{d}v = \int_0^u J_q^u(v)\mathrm{d}v \quad \text{and} \quad \int_0^x F(v)\mathrm{d}v \geq \int_0^x J_q^u(v)\mathrm{d}v \text{ for all } x \in [0, u]. \quad (5)$$

Many distributions satisfy Assumption A2, such as, for example, an arbitrary distribution on $[0, u]$ with sufficiently large mass points at 0 and u. Under Assumption A2, F is a mean-preserving spread of $J^u_{q_0}$, which implies that $q^* = q_0$ and $B^* = u$ in the unambiguous benchmark.[23] In general, we only know that $q^* \geq q_0$. Thanks to Assumption A2, the lower bound on revenue in Proposition 5(i) is $(1 - \alpha)q^*$, which corresponds exactly to the revenue that signal structure IP implements (cf. Subsection 5.1).

Assumption A3 is on the payoff-irrelevant states and strengthens Assumption A1 by adding a second state with full ambiguity. This enables us to construct signal structures that fully exploit combinations of ex ante optimism and interim pessimism or optimism.

Assumption A3. There are two states $\omega_0, \omega_1 \in \Omega$ such that for $\omega \in \{\omega_0, \omega_1\}$, $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega] = \mathbb{P}_{G_S}[\omega] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_G[\omega] = 1$.

We find that under Assumptions A2 and A3, the necessary conditions in Proposition 5 are also sufficient. This yields the following theorem, which fully characterizes all implementable outcomes. The proof is constructive, that is, for every outcome we define a corresponding signal structure in the appendix.

Theorem 2. *Assume A2 and A3. An outcome (U, R) with ex ante utility U and revenue R is implementable if and only if*

$$R \geq (1 - \alpha)q^* \quad \textit{and} \quad 0 \leq U \leq \alpha(\mathbb{E}_F[v] - R) + (1 - \alpha)\mathbb{E}_F[\max\{v - R, 0\}].$$

Note that the first condition represents a lower bound on R whereas the second condition represents both an upper bound on R and bounds on U. Figure 1 in Section 1 illustrates Theorem 2, assuming $\mathbb{P}_F[0] = \mathbb{P}_F[1] = 1/2$. The yellow and blue areas combined represent the set of all implementable outcomes, with the yellow area corresponding to the implementable outcomes without ambiguity as characterized in Proposition 1(ii). The left boundary of the set is given by $U = 0$, which ensures that the buyer's participation constraint is satisfied. The upper right boundary represents the Pareto frontier and is given by

$$U = \alpha(\mathbb{E}_F[v] - R) + (1 - \alpha)\mathbb{E}_F[\max\{v - R, 0\}]. \tag{6}$$

Compared to the frontier without ambiguity, which is given by $U = \mathbb{E}_F[v] - R$, ambiguity shifts the frontier outward. This is caused by ex ante optimism, al-

[23]The mean of J^B_q is increasing in q and decreasing in B. Among all J^B_q of which F is a mean-preserving spread, the lowest q corresponds to the highest possible B, i.e., $B = u$.

lowing for higher utility as the buyer anticipates in the best case to avoid buying when her valuation is below the price, whereas on the frontier without ambiguity she always buys and the price equals revenue. As the benefit of ex ante optimism increases with the price, the frontier corresponds to a (weakly) convex function.[24] Finally, the lower boundary is given by $R = (1-\alpha)q^*$, which is shifted downward compared to the lower boundary without ambiguity. This decrease in the lowest implementable revenue is caused by interim pessimism. By contrast, interim optimism combined with ex ante optimism permits to implement outcomes with revenue above $\mathbb{E}_F[v]$.

Comparing the two panels of Figure 1 conveys how changing ambiguity attitude α changes the set of implementable outcomes. Formally, as α decreases, the lower bound $(1-\alpha)q^*$ on R increases whereas the Pareto frontier given by (6) moves further outward as every R corresponds to a greater U. For high α as in the left panel, the effect of interim pessimism dominates, whereas for low α as in the right panel, the effects of ex ante and interim optimism dominate. As the buyer can benefit from both optimism and pessimism, she is in a similar situation in both cases. For the seller, however, a low α is preferable, as he only benefits from (the buyer's) optimism and suffers from pessimism.

6.3 Buyer- and Seller-Optimal Signal Structures

Let us now focus on the lower right and upper left corner of Figure 1: the buyer optimum and the seller optimum. We begin with the former. As is immediate from Theorem 2, the highest implementable ex ante utility U corresponds to the point on the Pareto frontier in Eq. (6) where revenue is at the lowest implementable level, that is, $R = (1-\alpha)q^*$.

Corollary 1. *Assume A2 and A3. Every buyer-optimal signal structure implements revenue* $(1-\alpha)q^*$*, price* $(1-\alpha)q^*$*, and ex ante utility*

$$U = \alpha(\mathbb{E}_F[v] - (1-\alpha)q^*) + (1-\alpha)\mathbb{E}_F[\max\{v - (1-\alpha)q^*, 0\}].$$

In Section 5, we described two improvements for the buyer's ex ante utility compared to the unambiguous benchmark of Proposition 1. Interim pessimism via signal structure IP reduces the seller's revenue and the corresponding decrease in price raises the buyer's utility (Proposition 2). Ex ante optimism via signal structure AO leaves the seller's revenue unchanged while increasing the buyer's

[24]In Figure 1 the frontier is linear as it is drawn for a binary prior.

utility (Proposition 3). The buyer-optimal outcome is obtained when combining both effects.

The buyer's ex ante utility is maximized by the signal structure H with signals $M = [q^*, B^*] \cup \{m_+, m_l, m_h\}$ where $B^* < m_+ < m_l < m_h$ and with distributions

$$H_{v,\omega}(m) = \begin{cases} J_{q^*}^{B^*}(m) & \text{if } \omega = \omega_0 \text{ and } v = 0, \\ \mathbf{1}(m \geq m_+) & \text{if } \omega = \omega_0 \text{ and } v > 0, \\ \mathbf{1}(m \geq m_l) & \text{if } \omega = \omega_1 \text{ and } v \leq (1-\alpha)q^*, \\ \mathbf{1}(m \geq m_h) & \text{if } \omega = \omega_1 \text{ and } v > (1-\alpha)q^*, \\ H_v^*(m) & \text{if } \omega \notin \{\omega_0, \omega_1\}. \end{cases}$$

In all states $\omega \notin \{\omega_0, \omega_1\}$, the signal structure coincides with the unambiguous benchmark in Proposition 1(i) with signals in $[q^*, B^*]$. In state ω_0, the signal structure induces interim pessimism like signal structure IP in Subsection 5.1, and in state ω_1, it induces ex ante optimism like signal structure AO in Subsection 5.2 by revealing whether the valuation is above $(1-\alpha)q^*$. From the seller's perspective, the signal is drawn from $\bar{H}^* = J_{q^*}^{B^*}$ on $[q^*, B^*]$, since ω_0 and ω_1 occur with probability zero under G_S. As under signal structure IP, the willingness to pay is $\hat{v}(m) = (1-\alpha)m$ for all $m \in [q^*, B^*]$, the price $(1-\alpha)q^*$ is thus optimal for the seller, and revenue $(1-\alpha)q^*$ is implemented. Given price $(1-\alpha)q^*$, the buyer buys for all signals except for m_l. From the ex ante perspective, the buyer buys if and only if her valuation is above the price in the best case (state ω_1), and she always buys in the worst case. Hence, the ex ante utility is as given in Corollary 1.

We now turn to the seller optimum. By Theorem 2, the highest implementable revenue corresponds to the point on the Pareto frontier (6) where $U = 0$.

Corollary 2. *Assume A2 and A3. Every seller-optimal signal structure implements revenue R^*, price R^*, and ex ante utility 0, where $R^* \in [\mathbb{E}_F[v], u]$ is (uniquely) defined by*

$$0 = \alpha(\mathbb{E}_F[v] - R^*) + (1-\alpha)\mathbb{E}_F[\max\{v - R^*, 0\}].$$

Whereas signal structure SE in Subsection 5.3 improves revenue relative to the unambiguous benchmark $\mathbb{E}_F[v]$, we now demonstrate how the highest implementable revenue can be attained by maximally exploiting interim and ex ante optimism. Consider the signal structure H with signals $M = \{m_0, m_l, m_h\}$ and distributions

$$H_{v,\omega}(m) = \begin{cases} \epsilon \mathbf{1}(m \geq m_l) + (1-\epsilon)\mathbf{1}(m \geq m_0) & \text{if } \omega = \omega_0 \text{ and } v < u, \\ \mathbf{1}(m \geq m_0) & \text{if } \omega = \omega_0 \text{ and } v = u, \\ \mathbf{1}(m \geq m_l) & \text{if } \omega = \omega_1 \text{ and } v \leq R^*, \\ \mathbf{1}(m \geq m_h) & \text{if } \omega = \omega_1 \text{ and } v > R^*, \\ \mathbf{1}(m \geq m_0) & \text{if } \omega \notin \{\omega_0, \omega_1\}, \end{cases}$$

where $\epsilon \in (0,1]$. In state ω_1, the signal structure induces ex ante optimism like signal structure *AO* in Subsection 5.2 by sending signal m_l for valuations below R^* and signal m_h for valuations above R^*. In every state $\omega \notin \{\omega_0, \omega_1\}$, it reveals no information and always sends signal m_0. In state ω_0, the signal structure sends signal m_0 for the highest valuation while randomizing between m_0 and m_l for all other valuations. Hence, signal m_0 implies an expected valuation of $\mathbb{E}_F[v]$ in the worst case and of $\mathbb{E}[v|m_0, \omega_0] \in (\mathbb{E}_F[v], u]$ in the best case (state ω_0).[25] This results in the willingness to pay $\hat{v}(m_0) = \alpha\mathbb{E}_F[v] + (1-\alpha)\mathbb{E}[v|m_0, \omega_0]$. Thus, there is interim optimism. Now, let ϵ be such that $\hat{v}(m_0) = R^*$. The seller expects signal m_0 to occur with probability one and, thus, optimally sets the price equal to $\hat{v}(m_0) = R^*$, resulting in revenue R^*. Given a price of R^*, the buyer buys for signals m_0 and m_h but not for signal m_l. From the ex ante perspective, the best case for the buyer is that she buys if and only if her valuation is above R^*, which corresponds to ω_1 occurring. The worst case for the buyer is that she always buys, which corresponds to any $\omega \notin \{\omega_0, \omega_1\}$ occurring. Hence, her ex ante utility is $\alpha(\mathbb{E}_F[v] - R^*) + (1-\alpha)\mathbb{E}_F[\max\{v - R^*, 0\}]$, which is equal to zero by the definition of R^*. Notice that signals are informative for the buyer in seller-optimal signal structures. By contrast, without ambiguity, uninformative signal structures are optimal for the seller. We next discuss dynamic consistency.

7 Dynamic Consistency

In general, models with ambiguity-sensitive preferences in dynamic settings cannot ensure consequentialism and dynamic consistency at the same time (see, e.g., Bleichrodt et al., 2021). Our model ensures consequentialism (i.e., the buyer's interim purchase decision only depends on the residual uncertainty and payoffs that are still possible) but permits dynamic inconsistency (i.e., the interim purchase

[25] Note that $\mathbb{E}[v|m_0, \omega_0] = ((1-\epsilon)(1-\mathbb{P}_F[u])\mathbb{E}_F[v|v < u] + \mathbb{P}_F[u]u)/((1-\epsilon)(1-\mathbb{P}_F[u]) + \mathbb{P}_F[u])$.

decision may differ from a contingent purchase plan the buyer deems optimal ex ante).[26] To define the ex ante utility in Section 3, we follow Siniscalchi's (2011) approach in dealing with this inconsistency by applying consistent planning (Strotz, 1955): ex ante the buyer correctly anticipates any preference reversals that determine her interim purchase decision. The same approach is often used by the literature on endogenously designed ambiguity, for example, by Bose and Renou (2014), Kellner and Le Quement (2018), and Beauchêne et al. (2019).

In this section, we demonstrate that the benefits of ambiguous product information do not hinge on dynamic inconsistency. In our model, whether the buyer's behavior is dynamically inconsistent depends on the signal structure. As we show, there are ambiguous signal structures that induce dynamically consistent behavior and make the buyer or the seller better off than any unambiguous signal structure.

To formally define dynamic consistency, let $a\colon M \to \{0,1\}$ denote a *purchase plan*, which specifies that the buyer buys after signal m if $a(m) = 1$ and does not buy if $a(m) = 0$. The *ex ante utility* from purchase plan a at price p is

$$U(a,p) = \alpha \min_{G\in\mathcal{G}} U_G(a,p) + (1-\alpha) \max_{G\in\mathcal{G}} U_G(a,p),$$

where

$$U_G(a,p) = \int_\Omega \int_0^u \int_M (v-p)a(m)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G(\omega).$$

On several occasions below, it will be useful to rewrite U_G using the definitions of conditional expectations and posteriors (see Footnotes 12 and 13) as

$$U_G(a,p) = \int_\Omega \int_M (\mathbb{E}[v|m,\omega] - p)a(m)\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G(\omega) \tag{7}$$

$$= \int_\Omega \int_M \left(\int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) - p \right) a(m)\mathrm{d}\bar{H}_{\omega'}(m)\mathrm{d}G(\omega'). \tag{8}$$

The buyer's purchase decision upon observing signal m corresponds to the purchase plan a^*_p with $a^*_p(m) = \mathbf{1}(\hat{v}(m) \geq p)$ for all $m \in M$. We say that the buyer's behavior is *dynamically consistent* at price p if a^*_p maximizes $U(a,p)$, that is, if the buyer finds her interim purchase decision also optimal from the ex ante perspective. A signal structure H *consistently implements* the outcome (U,R) of ex ante utility U and revenue R if H implements (U,R) and

$$U(a^*_p, p) \geq U(a,p) \quad \text{for all purchase plans } a \text{ and all prices } p \in [0,u].$$

[26] In experiments conducted by Dominiak et al. (2012) and Bleichrodt et al. (2021), the majority of ambiguity-averse subjects satisfied consequentialism and violated dynamic consistency.

That is, consistent implementation requires that the buyer's behavior is dynamically consistent at all prices, as the purchase decision at all prices determines the seller's pricing. An outcome is *consistently implementable*, if there is a signal structure that consistently implements it.

A sufficient condition for dynamically consistent behavior at all prices is that the buyer perceives no ambiguity about her valuation upon observing the signal. Formally, we say a signal structure induces *no relevant interim ambiguity after* signal m if $\int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) = \hat{v}(m)$ for all $G \in \mathcal{G}$. If this is the case, (8) reveals that the purchase decision $a(m) = a^*_p(m)$ maximizes $U_G(a,p)$ for any $G \in \mathcal{G}$, and thus it also maximizes the ex ante utility $U(a,p)$. An outcome is *implementable without relevant interim ambiguity* if it is implemented by a signal structure that induces no relevant interim ambiguity after each signal $m \in M$. Under such a signal structure the purchase plan $a = a^*_p$ maximizes $U(a,p)$, that is, the buyer's behavior is dynamically consistent, which yields part (i) of the following corollary.

Corollary 3.

(i) If an outcome (U,R) is implementable without relevant interim ambiguity, then it is consistently implementable.

(ii) Assume A1 with $\delta = 1$. An outcome (U,R) is implementable without relevant interim ambiguity if and only if

$$\mathbb{E}_F[v] \geq R \geq q^* \quad \textit{and} \quad 0 \leq U \leq \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\mathbb{E}_F[\max\{v - R, 0\}].$$

An example of a signal structure that induces no relevant interim ambiguity after each signal is *AO* in Subsection 5.2 on ex ante optimism (as each signal reveals whether state ω_0 occurred). By Corollary 3(i), *AO* consistently implements the outcome described in Proposition 3, and thus for all $\alpha < 1$, Theorem 1(i) remains valid even when restricting attention to signal structures that induce dynamically consistent behavior. Corollary 3(ii) characterizes all outcomes that are implementable without relevant interim ambiguity. These outcomes can be implemented by combining an unambiguous signal structure with ex ante optimism. They correspond to the subset of the implementable outcomes (U,R) in Theorem 2 where $R \in [q^*, \mathbb{E}_F[v]]$, that is, where the seller's revenue is in the range that is implementable by unambiguous signal structures. In Figure 4, the blue area indicates the outcomes that are implementable without relevant interim ambiguity, whereas the orange and gray areas indicate all other implementable outcomes in Theorem 2. Corollary 3 together with Proposition 5 implies that under A1 with

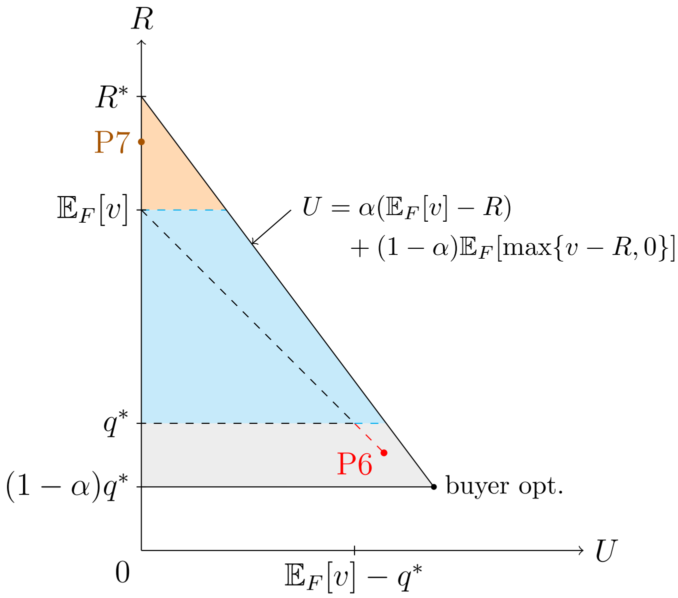

Figure 4: Each blue outcome is consistently implementable by Corollary 3, outcome P6 by Proposition 6(i), and outcome P7 by Proposition 7(i). Each orange outcome is consistently implementable in the case of Proposition 7(ii.b). The orange, blue, and gray areas represent all implementable outcomes as per Theorem 2.

$\delta = 1$, an outcome (U, R) with $R \in [q^*, \mathbb{E}_F[v]]$ is implementable if and only if it is consistently implementable.

Interim ambiguity is necessary for implementing revenue $R \notin [q^*, \mathbb{E}_F[v]]$, which relies on exploiting interim pessimism or optimism. Nevertheless, as we will show, both $R < q^*$ and $R > \mathbb{E}_F[v]$ are consistently implementable. We begin with $R < q^*$ and interim pessimism. Note that signal structure IP in Subsection 5.1 on interim pessimism does not necessarily induce dynamically consistent behavior: Under A1 with $\delta = 1$, IP implements revenue at the lower bound $(1-\alpha)q^*$, which equals zero if $\alpha = 1$. Revenue zero is implemented because $a_p^*(m) = 0$ for $m \in [q^*, B^*]$ for any price $p > 0$. This is dynamically inconsistent for small p, as the purchase plan $a(m) = 1$ for all m yields ex ante utility $\mathbb{E}_F[v] - p$ whereas a_p^* yields zero. We next show how under dynamic consistency the buyer may still benefit from interim pessimism.

Assume $\alpha = 1$, A1 with $\delta = 1$, and that the prior F has support $\{0, 1\}$ with $\mathbb{E}_F[v] = \mu$. Consider the signal structure H with $M = [\tilde{q}, \mu]$ where $\tilde{q} < q^*$ solves $\tilde{q}/\mu - \tilde{q}\ln(\tilde{q}/\mu) = \mu$.[27] In state ω_0, signal $m = \mu$ is drawn with probability one and $\mathbb{E}[v|m = \mu, \omega_0] = \mu$. In all states $\omega \neq \omega_0$, m is drawn from $\bar{H}_\omega = J_{\tilde{q}}^{\mu}$, and

[27] For F as here, q^* solves $\mathbb{E}_{J_{q^*}^1}[v] = q^* - q^*\ln(q^*) = \mu$. If $\mu = 0.5$, $\tilde{q} \approx 0.159$ and $q^* \approx 0.187$.

$\mathbb{E}[v|m,\omega] = m$ for $m < \mu$ but $\mathbb{E}[v|m = \mu, \omega] = 1$.[28] The ambiguity after signal $m = \mu$ results in $\hat{v}(\mu) = \min\{\mu, 1\} = \mu$. After signals $m < \mu$, H induces no relevant interim ambiguity, and thus to maximize $U(a,p)$ a purchase plan must satisfy $a(m) = a_p^*(m)$ for $m < \mu$. By (7), such plans a yield

$$U_G(a,p) = \mathbb{P}_G[\omega_0](\mu - p)a(\mu) \\ + (1 - \mathbb{P}_G[\omega_0]) \left(\int_{[\tilde{q},\mu)} \max\{m - p, 0\} \mathrm{d}J_{\tilde{q}}^{\mu}(m) + \mathbb{P}_{J_{\tilde{q}}^{\mu}}[\mu](1 - p)a(\mu) \right).$$

$\mathbb{P}_G[\omega_0] = 1$ minimizes U_G as in state ω_0 the signal is uninformative whereas in other states the buyer may obtain valuable information. Hence, $U(a,p) = (\mu - p)a(\mu)$, so that a_p^* maximizes $U(a,p)$ for all p. Since under the seller's prior $\hat{v}(m)$ is drawn from $J_{\tilde{q}}^{\mu}$, H consistently implements revenue $\tilde{q} < q^*$ and ex ante utility $\mu - \tilde{q}$. With part (i) of the following proposition we extend this result to all $\alpha \in (0,1]$ and almost all priors that satisfy A2.

Proposition 6. *Assume A1 with $\delta = 1$ and A2 with strict inequality in (5) for all $x \in (0,u)$.*

(i) For all $\alpha \in (0,1]$, there is a consistently implementable outcome (U,R) with $R < q^*$ and $U = \mathbb{E}_F[v] - R$.

(ii) There exist $0 < \alpha' \le \alpha'' < 1$ such that the following holds:

(ii.a) If $\alpha \in [0,\alpha']$, revenue $R = (1-\alpha)q^*$ is consistently implementable.

(ii.b) If $\alpha \in (\alpha'',1]$, revenue $R = (1-\alpha)q^*$ is not consistently implementable.

In Figure 4, the point P6 indicates a consistently implementable outcome as established by Proposition 6(i). In showing that interim pessimism can lower revenue and benefit the buyer, Proposition 6(i) is a dynamically consistent counterpart to Proposition 2. When restricting attention to signal structures that induce dynamically consistent behavior, Theorem 1(i) thus remains valid not only for $\alpha \in [0,1)$ as argued above but also for $\alpha = 1$ under the assumptions of Proposition 6(i). Recall from Theorem 2 that the lowest implementable revenue is $(1-\alpha)q^*$. Proposition 6(ii) shows that even revenue $(1-\alpha)q^*$ is consistently implementable if α is sufficiently close to 0 whereas this is impossible if α is sufficiently close to 1.[29]

We now turn to revenue $R > \mathbb{E}_F[v]$. Signal structure SE in Subsection 5.3 consistently implements such R, as we explain in the following. Assume $\alpha < 1$ and

[28] H is consistent with F as $\tilde{q}$ is defined so that $\int_{\tilde{q}}^1 \mathbb{E}[v|m,\omega]\mathrm{d}J_{\tilde{q}}^{\mu}(m) = \mu$.

[29] As seen from the proof, Proposition 6(ii.b) holds without Assumptions A1 and A2.

A1 with $\delta = 1$. As SE induces no relevant interim ambiguity after m_1 and m_2, we may restrict attention to plans a with $a(m) = a_p^*(m)$ for $m \neq m_3$ to maximize $U(a,p)$. Clearly, for $p \leq \mathbb{E}_F[v] - \sigma\epsilon$, also $a(m_3) = a_p^*(m_3) = 1$ maximizes $U(a,p)$. For $p > \mathbb{E}_F[v] - \sigma\epsilon$, (7) yields

$$\begin{aligned} U_G(a,p) &= \mathbb{P}_G[\omega_0](\mathbb{E}_F[v] + \epsilon - p)a(m_3)/2 \\ &\quad + (1 - \mathbb{P}_G[\omega_0])[\max\{\mathbb{E}_F[v] + \sigma\epsilon - p, 0\} + (\mathbb{E}_F[v] - \sigma\epsilon - p)a(m_3)]/2. \end{aligned} \tag{9}$$

Note that if $a(m_3) = 0$, $\mathbb{P}_G[\omega_0] = 1$ minimizes U_G whereas if $a(m_3) = 1$, $\mathbb{P}_G[\omega_0] = 0$ minimizes U_G. Hence, the plans a_0 with $a_0(m_3) = 0$ and a_1 with $a_1(m_3) = 1$ yield

$$\begin{aligned} U(a_0,p) &= (1-\alpha)\max\{\mathbb{E}_F[v] + \sigma\epsilon - p, 0\}/2, \\ U(a_1,p) &= \alpha\max\{\mathbb{E}_F[v] + \sigma\epsilon - p, 0\}/2 + (\mathbb{E}_F[v] + \sigma\epsilon - p)/2, \end{aligned}$$

where we used that $(1-\alpha)\epsilon - \alpha\sigma\epsilon = \sigma\epsilon$.[30] Observe that $U(a_1,p) \geq U(a_0,p)$ if and only if $p \leq \mathbb{E}_F[v] + \sigma\epsilon = \hat{v}(m_3)$. Consequently, $a = a_p^*$ maximizes $U(a,p)$ for all p, that is, SE consistently implements revenue $\mathbb{E}_F[v] + \sigma\epsilon > \mathbb{E}_F[v]$. As this generalizes to any $\delta \in (0,1]$, we obtain part (i) of the following proposition.

Proposition 7. *Assume A1.*

(i) For all $\alpha \in [0,1)$, revenue $R > \mathbb{E}_F[v]$ is consistently implementable.

(ii) Assume $\delta = 1$ and F has support $\{0,u\}$, that is, $\mathbb{P}_F[u] = 1 - F(0)$.

(ii.a) If $\alpha \geq 1 - [1-F(0)]/F(0)$, revenue R^* is consistently implementable.

(ii.b) If $\alpha \geq F(0)$, every outcome (U,R) with

$$R > \mathbb{E}_F[v] \text{ and } 0 \leq U \leq \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\mathbb{E}_F[\max\{v - R, 0\}]$$

is consistently implementable.

By Proposition 7(i), Theorem 1(ii) remains valid when restricting attention to signal structures that induce dynamically consistent behavior. Proposition 7(ii) provides additional results for the tractable case of priors with binary support and sufficiently large α: Part (ii.a) shows that even revenue R^* as implemented by seller-optimal signal structures (Corollary 2) is consistently implementable. Part (ii.b) together with Proposition 5 implies that an outcome (U,R) with $R >$

[30] Note that for a_1, the buyer ex ante perceives $\omega \neq \omega_0$ as the worst case and ω_0 as the best case, exactly like she does interim after signal m_3 when assessing her willingness to pay $\hat{v}(m_3)$.

$\mathbb{E}_F[v]$ is implementable if and only if it is consistently implementable. Notice that the restriction on α in (ii.a) is weaker than in (ii.b). In particular, (ii.a) holds for all $\alpha \in [0,1]$ if $F(0) \leq 1/2$. In Figure 4, the point P7 indicates a consistently implementable outcome as established by Proposition 7(i) whereas the orange area indicates the consistently implementable outcomes characterized by Proposition 7(ii.b).

If there is dynamic inconsistency, a decision maker may prefer not to receive freely available information.[31] Suppose learning is voluntary, that is, the buyer can choose ex ante to ignore the signal, in which case she buys if $\mathbb{E}_F[v] \geq p$. It is thus optimal for the buyer to follow the purchase plan a_p^{VL} such that $a_p^{VL}(m) = \mathbf{1}(\mathbb{E}_F[v] \geq p)$ for all m if $U(a_p^*, p) < \max\{\mathbb{E}_F[v] - p, 0\}$ and $a_p^{VL} = a_p^*$ otherwise. We say an outcome (U, R) is *implementable under voluntary learning* if there is a signal structure H that implements (U, R) and ensures $R \geq p \int_\Omega \int_M a_p^{VL}(m) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G_S(\omega)$ for all p, that is, the seller has no incentive to set a price that prompts the buyer to ignore the signal. Clearly, as dynamic consistency implies $a_p^{VL} = a_p^*$, every consistently implementable outcome is also implementable under voluntary learning. Yet, dynamic consistency is not necessary: for example, as revenue is at most $\mathbb{E}_F[v]$ when the buyer ignores the signal, every implementable outcome (U, R) with $R \geq \mathbb{E}_F[v]$ is also implementable under voluntary learning.

Finally, the results in this section further suggest that the benefits of ambiguous product information are not driven by the assumption that the buyer uses full Bayesian updating. Specifically, Corollary 3, Proposition 6(i), and Proposition 7 continue to hold under other 'consequentialist' updating rules that discard some of the priors based on the likelihood of the observed signal and update the other priors according to Bayes' rule, such as the maximum-likelihood rule (Gilboa and Schmeidler, 1993) or the rules proposed by Epstein and Schneider (2007) and Cheng (2022).[32]

[31] See Siniscalchi (2011) for the theoretical background and Kops and Pasichnichenko (2023) and Shishkin and Ortoleva (2023) for experimental evidence.

[32] If there is no relevant interim ambiguity after a signal under full Bayesian updating, the same is true under these other updating rules, so that Corollary 3 extends. Moreover, the proofs of Proposition 6(i) and Proposition 7 use signal structures where after each signal, there is either no relevant interim ambiguity or no prior is discarded as the signal has the same likelihood under all priors.

8 Conclusion

We have studied the welfare consequences of ambiguous product information in a model where a buyer with α-max-min preferences faces a price-setting seller. Ambiguity expands the set of possible outcomes in terms of ex ante utility for the buyer and revenue for the seller via two main channels. First, when making the purchase decision, ambiguity concerning the buyer's valuation can induce interim pessimism or interim optimism. This decreases or increases, respectively, price and revenue beyond the levels that are attainable without ambiguity. Second, at the outset, ambiguity about the future purchase decision can induce ex ante optimism, which moves the Pareto frontier outward. We have characterized all possible outcomes, showing that the buyer or the seller can benefit when the buyer is exposed to ambiguity.

As special cases of our analysis, we have also specified buyer-optimal and seller-optimal signal structures, which are expected to prevail if one side of the market fully controls the product information available to the buyer. If both the seller and the buyer (or a third party such as a consumer organization) partially control some of the information, the resulting ambiguous product information is included in our analysis but may depend on the details of the strategic interaction between the two sides. Whereas they may agree that ex ante optimism is beneficial, the seller's attempt to induce interim optimism would conflict with the buyer's preference for interim pessimism. For regulators of product information, our results suggest that promoting welfare calls for a nuanced approach: measures that reduce ambiguity can both increase or decrease welfare.

Appendix: Proofs

Proof of Proposition 1. Roesler and Szentes (2017) assume that the buyer's valuation is at most one. To transfer their results to our setting, consider the *normalized setting* where the valuation is $\tilde{v} = v/u$ and the prior is $\tilde{F}(\tilde{v}) = F(u\tilde{v})$. Clearly, ex ante utility U, revenue R, and price p are implemented by an unambiguous signal structure with $\bar{H}$ and $\mathbb{E}[v|m] = m$ if and only if $\tilde{U} = U/u$, $\tilde{R} = R/u$, and $\tilde{p} = p/u$ are implemented by an unambiguous signal structure with $\tilde{\bar{H}}(m) = \bar{H}(um)$ and $\mathbb{E}[\tilde{v}|m] = m$ in the normalized setting. Hence, part (i) of the proposition follows from Roesler and Szentes (2017, Theorem 1) and part (ii) from Roesler and Szentes (2017, pp. 2079–2080). □

Proof of Theorem 1. Part (i) follows from Propositions 1(i), 2, and 3. Part (ii) follows from Propositions 1(ii) and 4. □

Proof of Proposition 2. Fix any $\gamma \in [0, \mathbb{E}_F[v]/B^*]$ such that $F(\gamma q^*) > 0$. Let the set of possible signals be $M = [q^*, B^*] \cup \{m_+\}$. To be able to construct the signal structure $IP(\gamma)$ below, we first define the unambiguous signal structure H^0 with distributions H_v^0 such that

$$\bar{H}^0(m) = F(\gamma q^*) J_{q^*}^{B^*}(m) + \mathbf{1}(m \geq m_+)(1 - F(\gamma q^*)),$$

$$\mathbb{E}[v|m] = \begin{cases} \gamma m & \text{if } m \in [q^*, B^*], \\ \dfrac{1 - \gamma F(\gamma q^*)}{1 - F(\gamma q^*)} \mathbb{E}_F[v] & \text{if } m = m_+. \end{cases}$$

Note that $\mathbb{E}[v|m_+] > \mathbb{E}_F[v] \geq \gamma B^* \geq \mathbb{E}[v|m]$ for $m \neq m_+$ since $\gamma \leq E_F[v]/B^*$. Let $L(\tilde{v}) = \int_M \mathbf{1}(\mathbb{E}[v|m] \leq \tilde{v}) \mathrm{d}\bar{H}^0(m)$ denote the induced distribution of posterior means. We next show that the prior F is a mean-preserving spread of L, that is,

$$\int_0^u L(v)\mathrm{d}v = \int_0^u F(v)\mathrm{d}v \quad \text{and} \quad \int_0^x L(v)\mathrm{d}v \leq \int_0^x F(v)\mathrm{d}v \quad \text{for all } x \in [0, u]. \quad (10)$$

This ensures H_v^0 as defined above exist (see, e.g., Kolotilin, 2018). The equality in (10) is equivalent to $\mathbb{E}_L[v] = \mathbb{E}_F[v]$, which holds because

$$\mathbb{E}_L[v] = F(\gamma q^*)\gamma \int_{q^*}^{B^*} m \mathrm{d}J_{q^*}^{B^*}(m) + (1 - F(\gamma q^*))\mathbb{E}[v|m_+] = \mathbb{E}_F[v],$$

where we used $\int_{q^*}^{B^*} m \mathrm{d}J_{q^*}^{B^*}(m) = \mathbb{E}_F[v]$. As the equality holds, the inequality in (10) holds for $x \in [\mathbb{E}[v|m_+], u]$ because $L(v) = 1 \geq F(v)$ for $v \in [\mathbb{E}[v|m_+], u]$. On the other hand, the inequality in (10) also holds for $x \in [0, \mathbb{E}[v|m_+])$ because $L(v) = 0 \leq F(v)$ for $v \in [0, \gamma q^*]$ and $L(v) \leq F(\gamma q^*) \leq F(v)$ for $v \in (\gamma q^*, \mathbb{E}[v|m_+])$.

Now, consider the signal structure $IP(\gamma) = H$ with distributions

$$H_{v,\omega}(m) = \begin{cases} H_v^0(m) & \text{if } \omega = \omega_0, \\ H_v^*(m) & \text{if } \omega \neq \omega_0. \end{cases}$$

The signal structure IP in the main text corresponds to the special case where $\gamma = 0$. For all signals $m \in [q^*, B^*)$, the signal structure $IP(\gamma)$ admits densities

$\bar{h}_\omega(m) = j_{q^*}^{B^*}(m)$ for $\omega \neq \omega_0$ and $\bar{h}_{\omega_0}(m) = F(\gamma q^*) j_{q^*}^{B^*}(m)$. Hence, (1) yields

$$\begin{aligned} \mathbb{P}_{\hat{G}_m^G}[\omega_0] &= \frac{\mathbb{P}_G[\omega_0] F(\gamma q^*) j_{q^*}^{B^*}(m)}{\mathbb{P}_G[\omega_0] F(\gamma q^*) j_{q^*}^{B^*}(m) + (1 - \mathbb{P}_G[\omega_0]) j_{q^*}^{B^*}(m)} \\ &= \frac{\mathbb{P}_G[\omega_0] F(\gamma q^*)}{\mathbb{P}_G[\omega_0] F(\gamma q^*) + 1 - \mathbb{P}_G[\omega_0]}. \end{aligned} \tag{11}$$

Moreover, (11) also holds for signal $m = B^*$, which has positive probability under $\bar{H}_{\omega_0}$ and $\bar{H}_\omega$ such that $\int_{\{B^*\}} \mathrm{d}\bar{H}_{\omega_0}(m) = F(\gamma q^*) \int_{\{B^*\}} \mathrm{d}\bar{H}_\omega(m) = F(\gamma q^*) q^*/B^*$ for $\omega \neq \omega_0$. As $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = \delta$, we have $\min_{G \in \mathcal{G}} \mathbb{P}_{\hat{G}_m^G}[\omega_0] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_{\hat{G}_m^G}[\omega_0] = \tilde{\delta}$, where $\tilde{\delta} = \delta F(\gamma q^*)/[\delta F(\gamma q^*) + 1 - \delta]$. Therefore, for each signal $m \in [q^*, B^*]$, the buyer's willingness to pay is

$$\begin{aligned} \hat{v}(m) &= \alpha \min_{G \in \mathcal{G}} \left(\mathbb{P}_{\hat{G}_m^G}[\omega_0] \mathbb{E}[v|m, \omega_0] + (1 - \mathbb{P}_{\hat{G}_m^G}[\omega_0]) \mathbb{E}[v|m, \omega \neq \omega_0] \right) \\ &\quad + (1 - \alpha) \max_{G \in \mathcal{G}} \left(\mathbb{P}_{\hat{G}_m^G}[\omega_0] \mathbb{E}[v|m, \omega_0] + (1 - \mathbb{P}_{\hat{G}_m^G}[\omega_0]) \mathbb{E}[v|m, \omega \neq \omega_0] \right) \\ &= \alpha \min_{G \in \mathcal{G}} (1 - (1 - \gamma) \mathbb{P}_{\hat{G}_m^G}[\omega_0]) m + (1 - \alpha) \max_{G \in \mathcal{G}} (1 - (1 - \gamma) \mathbb{P}_{\hat{G}_m^G}[\omega_0]) m \\ &= \alpha (1 - (1 - \gamma) \tilde{\delta}) m + (1 - \alpha) m \\ &= \beta m \qquad \text{with } \beta = 1 - \alpha (1 - \gamma) \tilde{\delta} \in (0, 1). \end{aligned}$$

As m_+ is only in the support of $\bar{H}_{\omega_0}$, we have $\mathbb{P}_{\hat{G}_{m_+}^G}[\omega_0] = 1$ for all $G \in \mathcal{G}$, which implies $\hat{v}(m_+) = \mathbb{E}[v|m_+, \omega_0] > \mathbb{E}_F[v] \geq q^*$.

Under the seller's prior G_S, the state ω_0 has probability zero. As $\bar{H}_\omega = J_{q^*}^{B^*}$ for all $\omega \neq \omega_0$, the seller's revenue is

$$p \int_\Omega \int_M \mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G_S(\omega) = p \int_{q^*}^{B^*} \mathbf{1}(\beta m \geq p) \mathrm{d}J_{q^*}^{B^*}(m) = \beta q^*$$

for $p \in [\beta q^*, \beta B^*]$ and lower otherwise. The signal structure $IP(\gamma)$ thus implements price $p = \beta q^*$. At this price, the buyer always buys as $\hat{v}(m) \geq p$ for all $m \in M$. Therefore,

$$\int_0^u \int_M (v - p) \mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) = \int_0^u (v - p) \mathrm{d}F(v) = \mathbb{E}_F[v] - \beta q^*.$$

for all $\omega \in \Omega$. Hence, $IP(\gamma)$ implements ex ante utility $\mathbb{E}_F[v] - \beta q^* > \mathbb{E}_F[v] - q^*$ and revenue βq^*, where $\beta = 1 - \alpha(1 - \gamma) \delta F(\gamma q^*)/[\delta F(\gamma q^*) + 1 - \delta]$. □

Proof of Proposition 3. Consider the signal structure *AO* defined in the main text. As the support of $\bar{H}_\omega$ is $\{m_l, m_h\}$ if $\omega = \omega_0$ and $[q^*, B^*]$ if $\omega \neq \omega_0$, we have, for all $G \in \mathcal{G}$, $\mathbb{P}_{\hat{G}^G_m}[\omega_0] = 1$ if $m \in \{m_l, m_h\}$ and $\mathbb{P}_{\hat{G}^G_m}[\omega_0] = 0$ if $m \notin \{m_l, m_h\}$. Hence, the buyer perceives no relevant ambiguity when making the purchase decision, and we have

$$\hat{v}(m) = \begin{cases} m & \text{if } m \in [q^*, B^*], \\ \mathbb{E}_F[v|v < q^*] & \text{if } m = m_l, \\ \mathbb{E}_F[v|v \geq q^*] & \text{if } m = m_h. \end{cases}$$

Since $\bar{H}_\omega = J^{B^*}_{q^*}$ for all $\omega \neq \omega_0$ and $\mathbb{P}_{G_S}[\omega_0] = 0$, the seller's revenue is

$$p \int_\Omega \int_M \mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G_S(\omega) = p \int_{q^*}^{B^*} \mathbf{1}(m \geq p) \mathrm{d}J^{B^*}_{q^*}(m) = q^*$$

for $p \in [q^*, B^*]$ and lower otherwise, like in the unambiguous benchmark. Thus, signal structure AO implements price and revenue q^*. Given price q^*, the buyer buys if and only if $m \neq m_l$ because $\hat{v}(m_l) < q^* \leq \hat{v}(m)$ for all $m \neq m_l$. As signal m_l is sent if and only if $\omega = \omega_0$ and $v < q^*$, we have

$$\int_0^u \int_M (v - q^*) \mathbf{1}(\hat{v}(m) \geq q^*) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) = \begin{cases} \int_0^u \max\{v - q^*, 0\} \mathrm{d}F(v) & \text{if } \omega = \omega_0, \\ \int_0^u (v - q^*) \mathrm{d}F(v) & \text{if } \omega \neq \omega_0. \end{cases}$$

The buyer's ex ante assessment under prior G is thus

$$\begin{aligned} &\int_\Omega \int_0^u \int_M (v - q^*) \mathbf{1}(\hat{v}(m) \geq q^*) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) \mathrm{d}G(\omega) \\ &= \mathbb{P}_G[\omega_0] \mathbb{E}_F[\max\{v - q^*, 0\}] + (1 - \mathbb{P}_G[\omega_0]) \mathbb{E}_F[v - q^*], \end{aligned}$$

and since $\min_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = 0$ and $\max_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = \delta$, her ex ante utility is

$$\alpha \mathbb{E}_F[v - q^*] + (1 - \alpha)(\delta \mathbb{E}_F[\max\{v - q^*, 0\}] + (1 - \delta) \mathbb{E}_F[v - q^*]),$$

which is equivalent to the expression given in the proposition. □

Proof of Proposition 4. Consider the signal structure SE defined in the main text. We now generalize the analysis to arbitrary $\delta \in (0, 1]$. For any prior $G \in \mathcal{G}$, the posterior $\hat{G}^G_{m_3}$ after signal m_3 satisfies

$$\mathbb{P}_{\hat{G}^G_{m_3}}[\omega_0] = \frac{\mathbb{P}_G[\omega_0] \mathbb{P}_{\bar{H}_{\omega_0}}[m_3]}{\int_\Omega \mathbb{P}_{\bar{H}_\omega}[m_3] \mathrm{d}G(\omega)} = \mathbb{P}_G[\omega_0]$$

as $\int_\Omega \mathbb{P}_{\bar{H}_\omega}[m_3]\mathrm{d}G(\omega) = \mathbb{P}_{\bar{H}_{\omega_0}}[m_3] = 1/2$ for all $G \in \mathcal{G}$. Thus,

$$\int_\Omega \mathbb{E}[v|m_3,\omega]\mathrm{d}\hat{G}^G_{m_3}(\omega) = \mathbb{P}_G[\omega_0](\mathbb{E}_F[v]+\epsilon) + (1-\mathbb{P}_G[\omega_0])(\mathbb{E}_F[v]-\sigma\epsilon),$$

which implies willingness to pay

$$\hat{v}(m_3) = \mathbb{E}_F[v] - \alpha\sigma\epsilon + (1-\alpha)[\delta\epsilon - (1-\delta)\sigma\epsilon] = \mathbb{E}_F[v] + \sigma\epsilon.$$

In addition, $\hat{v}(m_1) = \mathbb{E}_F[v] - \epsilon$ and $\hat{v}(m_2) = \mathbb{E}_F[v] + \sigma\epsilon$. As under G_S, $\hat{v}(m) = \mathbb{E}_F[v] + \sigma\epsilon$ with probability 1, SE implements price $p = \mathbb{E}_F[v] + \sigma\epsilon$, revenue $p > \mathbb{E}_F[v]$, and ex ante utility

$$\alpha(\mathbb{E}_F[v]-p) + (1-\alpha)\left[\delta\frac{1}{2}(\mathbb{E}_F[v]+\epsilon-p) + (1-\delta)(\mathbb{E}_F[v]-p)\right] = 0. \qquad \square$$

Proof of Lemma 1. Let K be a feasible distribution of willingness to pay, that is, K satisfies (4) for some signal structure H. Since $\hat{v}(m) \in [0,u]$ for all m, $K(v) = 0$ if $v < 0$ and $K(v) = 1$ if $v \geq u$. Recall the definition of $\hat{v}(m)$ in (2). As $\mathbb{E}[v|m,\omega] \in [0,u]$ for all m and ω, we have

$$\hat{v}(m) \leq \alpha \min_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) + (1-\alpha)u \quad \text{and} \quad \min_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) \geq 0,$$

which implies

$$\alpha \min_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) \geq \max\{0, \hat{v}(m) - (1-\alpha)u\}.$$

Consequently,

$$\begin{aligned} \hat{v}(m) &\geq \max\{0,\hat{v}(m)-(1-\alpha)u\} + (1-\alpha)\max_{G\in\mathcal{G}} \int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^G_m(\omega) \\ &\geq \max\{0,\hat{v}(m)-(1-\alpha)u\} + (1-\alpha)\int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^{G_S}_m(\omega) \\ \iff \quad \min\{\hat{v}(m),(1-\alpha)u\} &\geq (1-\alpha)\int_\Omega \mathbb{E}[v|m,\omega]\mathrm{d}\hat{G}^{G_S}_m(\omega), \end{aligned} \tag{12}$$

where the second inequality uses $G_S \in \mathcal{G}$. By (4) and (12), the distribution K

satisfies

$$\begin{aligned}\int_0^u \min\{v,(1-\alpha)u\}\mathrm{d}K(v) &= \int_\Omega\int_M \min\{\hat{v}(m),(1-\alpha)u\}\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G_S(\omega)\\ &\geq (1-\alpha)\int_\Omega\int_M\int_\Omega \mathbb{E}[v|m,\omega']\mathrm{d}\hat{G}_m^{G_S}(\omega')\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G_S(\omega)\\ &= (1-\alpha)\int_\Omega\int_M \mathbb{E}[v|m,\omega]\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G_S(\omega)\\ &= (1-\alpha)\int_\Omega\int_0^u\int_M v\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G_S(\omega)\\ &= (1-\alpha)\mathbb{E}_F[v],\end{aligned}$$

where the second equality uses the definition of posteriors (see Footnote 13) and the third equality uses the definition of conditional expectations (see Footnote 12). Hence, $K \in \mathcal{K}$. □

Proof of Lemma 2. As the statement is clearly true if $\alpha = 1$, we assume $\alpha < 1$. We proceed in two steps. In Step 1, we show that the revenue R obtained under any $K \in \mathcal{K}$ is also obtained under a distribution $J_q^B \in \mathcal{K}$ with $q = R$. In Step 2, we show that $q = (1-\alpha)q_0$ is the smallest q such that $J_q^B \in \mathcal{K}$ for some B. By Step 1, any $K \in \mathcal{K}$ thus yields at least revenue $(1-\alpha)q_0$.

Step 1: Fix any $K \in \mathcal{K}$. Let p be an optimal price for the seller given distribution of willingness to pay K, and let $R = \int_p^u p dK(v)$ be the resulting revenue. Define the binary distribution $\tilde{F}$ on $\{0,u\}$ with $\tilde{F}(0) = 1 - \frac{1}{u}\int_0^u v\mathrm{d}K(v)$. Note that $\tilde{F}$ is a mean-preserving spread of K. Hence, (K,p) is an outcome in the model of Roesler and Szentes (2017) with prior $\tilde{F}$. By Roesler and Szentes (2017, Lemma 1), there thus exists a unique B such that K is a mean-preserving spread of J_R^B, and J_R^B yields revenue R. Since K is a mean-preserving spread of J_R^B, $J_R^B(v) = 0$ if $v < 0$, $J_R^B(v) = 1$ if $v \geq u$. Moreover,

$$\int_0^u \min\{v,(1-\alpha)u\}\mathrm{d}J_R^B(v) \geq \int_0^u \min\{v,(1-\alpha)u\}\mathrm{d}K(v)$$

as $\min\{v,(1-\alpha)u\}$ is concave in v. Consequently, $K \in \mathcal{K}$ implies $J_R^B \in \mathcal{K}$.

Step 2: Consider arbitrary $0 < q \leq B \leq u$. If $B \leq (1-\alpha)u$,

$$\begin{aligned}\int_0^u \min\{v,(1-\alpha)u\}\mathrm{d}J_q^B(v) = \int_q^B v\mathrm{d}J_q^B(v) = \int_q^B \frac{q}{v}\mathrm{d}v + q\\ = q\ln(B) - q\ln(q) + q,\end{aligned}$$

and if $B > (1-\alpha)u$,

$$\int_0^u \min\{v, (1-\alpha)u\} \mathrm{d}J_q^B(v) = \int_q^{(1-\alpha)u} \frac{q}{v} \mathrm{d}v + [1 - J_q^B((1-\alpha)u)](1-\alpha)u$$
$$= q\ln((1-\alpha)u)) - q\ln(q) + q.$$

Hence, $J_q^B \in \mathcal{K}$ if and only if $0 < q \leq B \leq u$ and

$$q\ln(\min\{B, (1-\alpha)u\}) - q\ln(q) + q \geq (1-\alpha)\mathbb{E}_F[v]. \tag{13}$$

Note that the left-hand side of (13) is increasing in q and B. The smallest q obtains when (13) holds with equality and $B \geq (1-\alpha)u$. That is, the smallest q solves $q\ln((1-\alpha)u) - q\ln(q) + q = (1-\alpha)\mathbb{E}_F[v]$, which is equivalent to $q = (1-\alpha)q_0$. $\square$

Proof of Proposition 5. As explained in the main text, part (i) directly follows from Lemmas 1 and 2. We are left to prove part (ii). From the ex ante perspective, the buyer's pessimistic assessment satisfies

$$\begin{aligned}
&\min_{G\in\mathcal{G}} \int_\Omega \int_0^u \int_M (v-p)\mathbf{1}(\hat{v}(m) \geq p)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G(\omega) \\
&\quad\leq \int_\Omega \int_0^u \int_M (v-p)\mathbf{1}(\hat{v}(m) \geq p)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G_S(\omega) \\
&\quad\leq \int_\Omega \int_0^u \int_M v\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G_S(\omega) - R \\
&\quad= \mathbb{E}_F[v] - R
\end{aligned}$$

where the first inequality uses $G_S \in \mathcal{G}$. The buyer's optimistic assessment satisfies

$$\max_{G\in\mathcal{G}} \int_\Omega \int_0^u \int_M (v-p)\mathbf{1}(\hat{v}(m) \geq p)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G(\omega) \leq \int_0^u \max\{v-R, 0\}\mathrm{d}F(v)$$

because $(v-p)\mathbf{1}(\hat{v}(m) \geq p) \leq \max\{v-p, 0\} \leq \max\{v-R, 0\}$, as $R \leq p$ by definition. Consequently, $U \leq \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\mathbb{E}_F[\max\{v-R, 0\}]$. $\square$

Proof of Theorem 2. Let $\bar{U}(R) = \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\mathbb{E}_F[\max\{v-R, 0\}]$. We will show that all (U, R) such that $(1-\alpha)q^* \leq R$ and $0 \leq U \leq \bar{U}(R)$ are implementable. To do so, we split this set of (U, R) into the four regions labeled I, II, III, and IV in Figure 5. We will consider each region in turn and construct signal structures that implement all (U, R) within that region. Outcomes outside the four regions are not implementable: $R < (1-\alpha)q^*$ is impossible by Proposition 5(i), $\bar{U}(R) < U$ is impossible by Proposition 5(ii), and $U < 0$ violates the buyer's

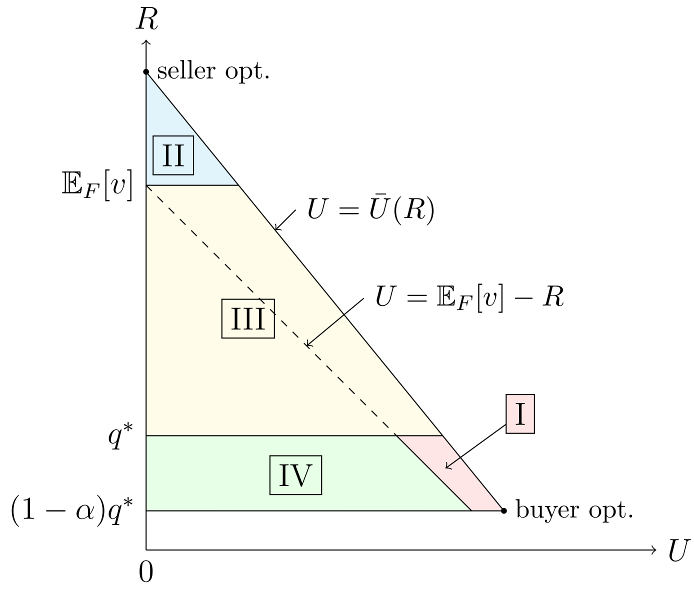

Figure 5: The four regions that are treated separately in the proof of Theorem 2.

participation constraint.

Region I: $(1-\alpha)q^* \leq R \leq q^*$ **and** $\mathbb{E}_F[v] - R \leq U \leq \bar{U}(R)$**.** We begin with defining three unambiguous signal structures. First, let H^* with signals $M^* = [q^*, u]$ and distributions H_v^* be a buyer-optimal unambiguous signal structure as characterized by Proposition 1(i). Most importantly, we thus have $\mathbb{E}[v|m] = m$ and $\bar{H}^* = J_{q^*}^u$, where $B^* = u$ because of Assumption A2. Second, define the unambiguous signal structure H^{IP} with signals $M^{IP} = [q^*, u] \cup \{m_+\}$ for some $m_+ \in (u, 2u)$ and distributions

$$H_v^{IP}(m) = \begin{cases} J_{q^*}^u(m) & \text{if } v = 0, \\ \mathbf{1}(m \geq m_+) & \text{if } v > 0. \end{cases}$$

This signal structure reveals whether $v = 0$ (signals $m \in [q^*, u]$) or not (signal m_+). Finally, define the unambiguous signal structure H^{AO} with signals $M^{AO} = [2u, 3u]$ and distributions

$$H_v^{AO}(m) = \mathbf{1}(m \geq 2u + v).$$

This signal structure perfectly reveals the valuation: it is $v = m - 2u$ for any m.

Now, we define the family of signal structures $T_I(\epsilon_{IP}, \epsilon_{AO}) = H$ with parameters $\epsilon_{IP}, \epsilon_{AO} \in [0,1]$, signals $M = [q^*, u] \cup \{m_+\} \cup [2u, 3u]$, and distributions

$$H_{v,\omega}(m) = \begin{cases} \epsilon_{IP} H_v^{IP}(m) + (1-\epsilon_{IP}) H_v^*(m) & \text{if } \omega = \omega_0, \\ \epsilon_{AO} H_v^{AO}(m) + (1-\epsilon_{AO}) H_v^*(m) & \text{if } \omega = \omega_1, \\ H_v^*(m) & \text{if } \omega \notin \{\omega_0, \omega_1\}. \end{cases}$$

The parameters ϵ_{IP} and ϵ_{AO} control the extent of interim pessimism and ex ante optimism, respectively. Under signal structure T_I, the conditional expectations are

$$\mathbb{E}[v|m,\omega_0] = \begin{cases} \dfrac{\epsilon_{IP} F(0) j_{q^*}^u(m) 0 + (1-\epsilon_{IP}) \bar{h}^*(m) m}{\epsilon_{IP} F(0) j_{q^*}^u(m) + (1-\epsilon_{IP}) \bar{h}^*(m)} & \text{if } m \in [q^*, u), \\ \dfrac{\epsilon_{IP} F(0) \mathbb{P}_{J_{q^*}^u}[u] 0 + (1-\epsilon_{IP}) \mathbb{P}_{\bar{H}^*}[u] u}{\epsilon_{IP} F(0) \mathbb{P}_{J_{q^*}^u}[u] + (1-\epsilon_{IP}) \mathbb{P}_{\bar{H}^*}[u]} & \text{if } m = u, \\ \mathbb{E}[v|v>0] & \text{if } m = m_+ \end{cases}$$

$$= \begin{cases} \dfrac{1-\epsilon_{IP}}{\epsilon_{IP} F(0) + 1 - \epsilon_{IP}} m & \text{if } m \in [q^*, u], \\ \mathbb{E}[v|v>0] & \text{if } m = m_+, \end{cases}$$

$$\mathbb{E}[v|m,\omega_1] = \begin{cases} m & \text{if } m \in [q^*, u], \\ m - 2u & \text{if } m \in [2u, 3u], \end{cases}$$

and $\mathbb{E}[v|m,\omega] = m$ for $\omega \notin \{\omega_0, \omega_1\}$.

As m_+ and $[2u, 3u]$, respectively, is only in the support of $\bar{H}_{\omega_0}$ and $\bar{H}_{\omega_1}$, we have $\mathbb{P}_{\hat{G}_m^G}[\omega_0] = 1$ if $m = m_+$ and $\mathbb{P}_{\hat{G}_m^G}[\omega_1] = 1$ if $m \in [2u, 3u]$ for all $G \in \mathcal{G}$. For $m \in [q^*, u]$,

$$\mathbb{P}_{\hat{G}_m^G}[\omega_0] = \frac{(\epsilon_{IP} F(0) + 1 - \epsilon_{IP}) \mathbb{P}_G[\omega_0]}{(\epsilon_{IP} F(0) + 1 - \epsilon_{IP}) \mathbb{P}_G[\omega_0] + (1-\epsilon_{AO}) \mathbb{P}_G[\omega_1] + \mathbb{P}_G[\omega \notin \{\omega_0, \omega_1\}]}$$

for all $G \in \mathcal{G}$, so that $\max_{G \in \mathcal{G}} \mathbb{P}_{\hat{G}_m^G}[\omega_0] = \max_{G \in \mathcal{G}} \mathbb{P}_G[\omega_0] = 1$ and $\min_{G \in \mathcal{G}} \mathbb{P}_{\hat{G}_m^G}[\omega_0] = \mathbb{P}_{G_S}[\omega_0] = 0$. Then, for $m \in [q^*, u]$, the willingness to pay is

$$\hat{v}(m) = \alpha \mathbb{E}[v|m,\omega_0] + (1-\alpha) \mathbb{E}[v|m, \omega \neq \omega_0] = \left(1 - \alpha \frac{\epsilon_{IP} F(0)}{\epsilon_{IP} F(0) + 1 - \epsilon_{IP}}\right) m,$$

whereas $\hat{v}(m_+) = \mathbb{E}_F[v|v>0]$ and $\hat{v}(m) = m - 2u$ for $m \in [2u, 3u]$.

Since $\bar{H}_\omega = J_{q^*}^u$ for all $\omega \notin \{\omega_0, \omega_1\}$ and $\mathbb{P}_{G_S}[\omega_0] = \mathbb{P}_{G_S}[\omega_1] = 0$, from the seller's perspective the willingness to pay $y = \hat{v}(m) = \kappa m$ is drawn from $J_{\kappa q^*}^{\kappa u}(y)$

where

$$\kappa = 1 - \alpha \frac{\epsilon_{IP} F(0)}{\epsilon_{IP} F(0) + 1 - \epsilon_{IP}}.$$

Thus, the seller's revenue is

$$p \int_{\Omega} \int_{M} \mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}_{\omega}(m) \mathrm{d}G_S(\omega) = p \int_{q^*}^{u} \mathbf{1}(\kappa m \geq p) \mathrm{d}J_{q^*}^{u}(m) = \kappa q^*$$

for $p \in [\kappa q^*, \kappa u]$ and lower otherwise. Accordingly, the signal structure $T_I(\epsilon_{IP}, \epsilon_{AO})$ implements price $p = \kappa q^*$ and revenue $R = \kappa q^*$. In particular, each $R \in [(1-\alpha)q^*, q^*]$ is implemented for some $\epsilon_{IP} \in [0,1]$.

Next, we fix $\epsilon_{IP} \in [0,1]$, which fixes $R = p = \kappa q^*$. Note that $\hat{v}(m) \geq p$ for all $m \in [q^*, u] \cup \{m_+\}$, so that from the ex ante perspective the buyer expects to always buy in every state $\omega \neq \omega_1$, resulting in expected payoff $\mathbb{E}_F[v] - R$. In state ω_1, however, the buyer expects with probability ϵ_{AO} to observe a signal $m \in [2u, 3u]$ where $\hat{v}(m) = v$. These signals allow her to buy if and only if $v \geq R$, resulting in expected payoff $\epsilon_{AO}\mathbb{E}_F[\max\{v - R, 0\}] + (1 - \epsilon_{AO})(\mathbb{E}_F[v] - R)$. Hence, $T_I(\epsilon_{IP}, \epsilon_{AO})$ implements ex ante utility

$$U = (1 - (1-\alpha)\epsilon_{AO})(\mathbb{E}_F[v] - R) + (1-\alpha)\epsilon_{AO}\mathbb{E}_F[\max\{v - R, 0\}]$$

because

$$\begin{aligned}
&\min_{G \in \mathcal{G}} \int_{\Omega} \int_0^u \int_M (v - R)\mathbf{1}(\hat{v}(m) \geq R)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G(\omega) = \mathbb{E}_F[v] - R, \\
&\max_{G \in \mathcal{G}} \int_{\Omega} \int_0^u \int_M (v - R)\mathbf{1}(\hat{v}(m) \geq R)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v)\mathrm{d}G(\omega) \\
&\qquad = \epsilon_{AO}\mathbb{E}_F[\max\{v - R, 0\}] + (1 - \epsilon_{AO})(\mathbb{E}_F[v] - R).
\end{aligned}$$

In particular, each $U \in [\mathbb{E}_F[v] - R, \bar{U}(R)]$ is implemented for some $\epsilon_{AO} \in [0,1]$.

Region II: $\mathbb{E}_F[v] \leq R$ and $0 \leq U \leq \bar{U}(R)$. We first define another unambiguous signal structure, namely H^{IO} with two signals $m_1 < m_2 < u$ and distributions

$$H_v^{IO}(m) = \begin{cases} \mathbf{1}(m \geq m_1) & \text{if } v < u, \\ \mathbf{1}(m \geq m_2) & \text{if } v = u. \end{cases}$$

This signal structure reveals whether $v < u$ (signal m_1) or $v = u$ (signal m_2). Next, we define the family of signal structures $T_{II}(\epsilon_{IO}, \epsilon_{AO}) = H$ with parameters

$\epsilon_{IO}, \epsilon_{AO} \in [0,1]$, signals $M = \{m_1, m_2\} \cup [2u, 3u]$, and distributions

$$H_{v,\omega}(m) = \begin{cases} \epsilon_{IO} H_v^{IO}(m) + (1-\epsilon_{IO})\mathbf{1}(m \geq m_2) & \text{if } \omega = \omega_0, \\ \epsilon_{AO} H_v^{AO}(m) + (1-\epsilon_{AO})\mathbf{1}(m \geq m_2) & \text{if } \omega = \omega_1, \\ \mathbf{1}(m \geq m_2) & \text{if } \omega \notin \{\omega_0, \omega_1\}. \end{cases}$$

The parameters ϵ_{IO} and ϵ_{AO} control the extent of interim and ex ante optimism, respectively. Under signal structure T_{II}, we have $\mathbb{E}[v|m_2, \omega] = \mathbb{E}_F[v]$ for $\omega \neq \omega_0$ and

$$\mathbb{E}[v|m_2, \omega_0] = \frac{\epsilon_{IO}\mathbb{P}_F[u]u + (1-\epsilon_{IO})\mathbb{E}_F[v]}{\epsilon_{IO}\mathbb{P}_F[u] + 1 - \epsilon_{IO}}.$$

Moreover, both $\mathbb{P}_{\hat{G}^G_{m_2}}[\omega_0] = 1$ and $\mathbb{P}_{\hat{G}^G_{m_2}}[\omega_0] = 0$ for some $G \in \mathcal{G}$. Hence,

$$\hat{v}(m_2) = \alpha\mathbb{E}_F[v] + (1-\alpha)\mathbb{E}[v|m_2, \omega_0] = \mathbb{E}_F[v] + (1-\alpha)\frac{\epsilon_{IO}\mathbb{P}_F[u](u - \mathbb{E}_F[v])}{\epsilon_{IO}\mathbb{P}_F[u] + 1 - \epsilon_{IO}}. \quad (14)$$

Since signal m_2 has probability 1 under the seller's prior G_S, T_{II} implements price $p = \hat{v}(m_2)$ and revenue $R = \hat{v}(m_2)$. In particular, each $R \geq \mathbb{E}_F[v]$ such that $\bar{U}(R) \geq 0$ is implemented for some $\epsilon_{IO} \in [0,1]$. To see this, note that $R = \hat{v}(m_2)$ is continuous in ϵ_{IO}, that $R = \mathbb{E}_F[v]$ and $\bar{U}(R) > 0$ if $\epsilon_{IO} = 0$, and that $R = \mathbb{E}_F[v] + (1-\alpha)(u - \mathbb{E}_F[v])$ and $\bar{U}(R) = (1-\alpha)\mathbb{E}_F[\max\{v-u, \alpha(\mathbb{E}_F[v]-u)\}] < 0$ if $\epsilon_{IO} = 1$.

Consider any ϵ_{IO} such that $T_{II}(\epsilon_{IO}, \epsilon_{AO})$ implements R with $\bar{U}(R) \geq 0$. We are left to show that each $U \in [0, \bar{U}(R)]$ is implemented for some ϵ_{AO}. Note that after signal m_1 the buyer does not buy because $\hat{v}(m_1) = \mathbb{E}_F[v|v < u] < \mathbb{E}_F[v] \leq R$, whereas after any signal $m \in [2u, 3u]$ she buys if and only if $v \geq R$ because $\hat{v}(m) = v$. Let $\tilde{U}(\omega)$ denote the ex ante expected payoff in state ω, that is,

$$\tilde{U}(\omega) = \int_0^u \int_M (v - R)\mathbf{1}(\hat{v}(m) \geq R)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v).$$

Then, we have $\tilde{U}(\omega) = \mathbb{E}_F[v] - R \leq 0$ for all $\omega \notin \{\omega_0, \omega_1\}$ whereas

$$\tilde{U}(\omega_0) = \epsilon_{IO}\mathbb{P}_F[u](u - R) + (1-\epsilon_{IO})(\mathbb{E}_F[v] - R) \geq \mathbb{E}_F[v] - R,$$
$$\tilde{U}(\omega_1) = \epsilon_{AO}\mathbb{E}_F[\max\{v - R, 0\}] + (1-\epsilon_{AO})(\mathbb{E}_F[v] - R) \geq \mathbb{E}_F[v] - R.$$

Hence, the implemented ex ante utility is

$$U = \alpha(\mathbb{E}_F[v] - R) + (1-\alpha)\max\{\tilde{U}(\omega_0), \tilde{U}(\omega_1)\},$$

which is continuous in ϵ_{AO}. Moreover, $\epsilon_{AO}=1$ yields $U=\bar{U}(R)$, whereas – using Eq. (14) for the second equality – it is easily verified that $\epsilon_{AO}=0$ yields

$$U=\alpha(\mathbb{E}_F[v]-R)+(1-\alpha)\tilde{U}(\omega_0)=-\alpha(1-\alpha)(u-\mathbb{E}_F[v])\frac{\epsilon_{IO}^2\mathbb{P}_F[u](1-\mathbb{P}_F[u])}{\epsilon_{IO}\mathbb{P}_F[u]+1-\epsilon_{IO}}\leq 0.$$

Thus, each $U\in[0,\bar{U}(R)]$ is implemented for some $\epsilon_{AO}\in[0,1]$.

Region III: $q^*\leq R\leq\mathbb{E}_F[v]$ **and** $0\leq U\leq\bar{U}(R)$**.** Fix any $q\in[q^*,\mathbb{E}_F[v]]$ and let $B\in[\mathbb{E}_F[v],u]$ be such that $\mathbb{E}_{J_q^B}[v]=\mathbb{E}_F[v]$. Note that $J_{q^*}^u$ is a mean-preserving spread of J_q^B, which implies that the prior F is a mean-preserving spread of J_q^B. Hence, there is an unambiguous signal structure H° with signals $M^\circ=[q,B]$ and distributions H_v° such that $\mathbb{E}[v|m]=m$ and $\bar{H}^\circ=J_q^B$.

Define the family of signal structures $T_{III}(q,\epsilon_{AO})=H$ with parameters $q\in[q^*,\mathbb{E}_F[v]]$ and $\epsilon_{AO}\in[0,1]$, signals $M=[q,B]\cup[2u,3u]$, and distributions

$$H_{v,\omega}(m)=\begin{cases}\epsilon_{AO}H_v^{AO}(m)+(1-\epsilon_{AO})H_v^\circ(m) & \text{if } \omega=\omega_0,\\ H_v^\circ(m) & \text{if } \omega\neq\omega_0.\end{cases}$$

Then, it is straightforward to verify that $\hat{v}(m)=\mathbb{E}[v|m,\omega]=m$ for all $m\in[q,B]$ and $\omega\in\Omega$ whereas $\hat{v}(m)=\mathbb{E}[v|m,\omega_0]=m-2u$ for $m\in[2u,3u]$.

As $\mathbb{P}_{G_S}[\omega_0]=0$, from the seller's perspective $y=\hat{v}(m)$ is drawn from $\bar{H}^\circ(y)=J_q^B(y)$. Hence, the signal structure $T_{III}(q,\epsilon_{AO})$ implements revenue $R=q$ and every price $p\in[q,B]$. Thus, each $R\in[q^*,\mathbb{E}_F[v]]$ is implemented by $T_{III}(R,\epsilon_{AO})$.

Consider $T_{III}(R,0)$ and any implemented price $p\in[R,B]$. Then the signal structure is unambiguous and the implemented ex ante utility is

$$U=\int_0^u\int_R^B(v-p)\mathbf{1}(\hat{v}(m)\geq p)\mathrm{d}H_v^\circ(m)\mathrm{d}F(v)=\int_p^B(m-p)\mathrm{d}\bar{H}^\circ(m).$$

Hence, each $U\in[0,\mathbb{E}_F[v]-R]$ is implemented for some $p\in[R,B]$.

Now, consider $T_{III}(R,\epsilon_{AO})$ and implemented price $p=R$. Then ex ante the buyer expects in state ω_0 with probability ϵ_{AO} to buy if and only if $v\geq R$, whereas she expects to always buy in all other cases. The implemented ex ante utility is thus

$$U=\alpha(\mathbb{E}_F[v]-R)+(1-\alpha)[\epsilon_{AO}\mathbb{E}[\max\{v-R,0\}]+(1-\epsilon_{AO})(\mathbb{E}_F[v]-R)].$$

In particular, each $U\in[\mathbb{E}_F[v]-R,\bar{U}(R)]$ is implemented for some $\epsilon_{AO}\in[0,1]$.

Region IV: $(1-\alpha)q^* \le R < q^*$ **and** $0 \le U \le \mathbb{E}_F[v] - R$**.** Define the family of signal structures $T_{IV}(\epsilon) = H$ with parameter $\epsilon \in (0,1]$, set of signals $M = [q^*, u+1] \cup \{m_+\}$ where $m_+ > u+1$, and distributions

$$H_{v,\omega}(m) = \begin{cases} H_v^X(m) & \text{if } \omega = \omega_0, \\ H_v^{**}(m) & \text{if } \omega \neq \omega_0. \end{cases}$$

Both H_v^X and H_v^{**} depend on ϵ and will be defined below. Recall that the distributions H_v^* ensure $\bar{H}^* = J_{q^*}^u$ and $\mathbb{E}[v|m] = m$. Clearly, $H_v^*(u) = 1$ for all v. Moreover, as $J_{q^*}^u$ has an atom of size q^*/u at $m = u$, we must have $\mathbb{P}_{H_v^*}[u] = 0$ for all $v < u$ and $\mathbb{P}_{\bar{H}^*}[u] = \mathbb{P}_F[u]\mathbb{P}_{H_u^*}[u] = q^*/u$. To simplify the notation, define

$$\xi = \mathbb{P}_{H_u^*}[u] = \frac{q^*}{u\mathbb{P}_F[u]}.$$

The distributions H_v^{**} coincide with H_v^* for all v except that the atom of H_u^* at signal $m = u$ is replaced by a distribution of signals $m \in [u, u+1]$. That is,

$$H_v^{**}(m) = \begin{cases} H_v^*(m) & \text{if } v < u \text{ or } m < u, \\ \dfrac{J^u_{\kappa(u)q^*}(\kappa(m)u) - 1 + \mathbb{P}_F[u]}{\mathbb{P}_F[u]} & \text{if } v = u \text{ and } m \in [u, u+1], \\ 1 & \text{if } v = u \text{ and } m > u+1, \end{cases}$$

where the continuous function $\kappa\colon [u, u+1] \to [1-\alpha, 1]$ is defined by

$$\kappa(m) = 1 - \alpha \frac{\mathbb{P}_F[0](u+1-m)\epsilon\xi}{2\epsilon(m-u) + (1-\epsilon)\xi + \mathbb{P}_F[0](u+1-m)\epsilon\xi}.$$

Note that for any $\epsilon \in (0,1]$, κ satisfies $1-\alpha \le \kappa(u) < \kappa(u+1) = 1$ and is strictly increasing. Note also that $H_u^{**}(u+1) = 1$ and $H_u^{**}(u) = 1 - \xi$, that is, H_u^{**} is continuous at u. For the unconditional distribution of signals $\bar{H}^{**}$, we then have

$$\bar{H}^{**}(m) = \begin{cases} \bar{H}^*(m) = J_{q^*}^u(m) & \text{if } m \in [q^*, u), \\ 1 - \mathbb{P}_F[u] + \mathbb{P}_F[u]\frac{J^u_{\kappa(u)q^*}(\kappa(m)u) - 1 + \mathbb{P}_F[u]}{\mathbb{P}_F[u]} = J^u_{\kappa(u)q^*}(\kappa(m)u) & \text{if } m \in [u, u+1]. \end{cases}$$

In particular, $\bar{H}^{**}(u+1)=1$ and $\bar{H}^{**}(u)=J^u_{\kappa(u)q^*}(\kappa(u)u)=1-q^*/u$, so that $\bar{H}^{**}$ is continuous at $m=u$. For $\omega\neq\omega_0$, signal m yields the expected valuation

$$\mathbb{E}[v|m,\omega]=\begin{cases} m & \text{if } m\in[q^*,u),\\ u & \text{if } m\in[u,u+1]. \end{cases} \tag{15}$$

The distributions H^X_v are defined as

$$H^X_v(m)=\begin{cases} (1-\epsilon)H^*_0(m)+\epsilon J^u_{q^*}(m) & \text{if } v=0 \text{ and } m<u,\\ 1-[1+(u+1-m)^2]\epsilon q^*/(2u) & \text{if } v=0 \text{ and } m\in[u,u+1],\\ 1-\mathbf{1}(m<m_+)\epsilon q^*/(2u) & \text{if } v=0 \text{ and } m>u+1,\\ (1-\epsilon)H^*_v(m)+\epsilon\mathbf{1}(m\geq m_+) & \text{if } v\in(0,u),\\ (1-\epsilon)H^*_u(m) & \text{if } v=u \text{ and } m<u,\\ (1-\epsilon)(1-\xi)+(1-\epsilon)\xi(m-u)+\epsilon(m-u)^2 & \text{if } v=u \text{ and } m\in[u,u+1],\\ 1 & \text{if } v=u \text{ and } m>u+1. \end{cases}$$

For the unconditional distribution of signals $\bar{H}^X$, we then have

$$\bar{H}^X(m)=\begin{cases} (1-\epsilon)\bar{H}^*(m)+\epsilon\mathbb{P}_F[0]J^u_{q^*}(m) & \text{if } m<u\\ \mathbb{P}_F[0](1-[1+(u+1-m)^2]\epsilon q^*/(2u))+(1-\epsilon)\mathbb{P}_F[v\in(0,u)] & \\ +\mathbb{P}_F[u][(1-\epsilon)(1-\xi)+(1-\epsilon)\xi(m-u)+\epsilon(m-u)^2] & \text{if } m\in[u,u+1],\\ 1-\epsilon\mathbf{1}(m<m_+)(\mathbb{P}_F[0]q^*/(2u)+\mathbb{P}_F[v\in(0,u)]) & \text{if } m>u+1, \end{cases}$$

which is easily verified to be continuous on $[q^*,u+1]$. The expected valuation upon signal m is for $m\in[q^*,u)$,

$$\mathbb{E}[v|m,\omega_0]=\frac{(1-\epsilon)\bar{h}^*(m)m}{(1-\epsilon)\bar{h}^*(m)+\mathbb{P}_F[0]\epsilon j^u_{q^*}(m)}=\frac{1-\epsilon}{1-\epsilon+\mathbb{P}_F[0]\epsilon}m \tag{16}$$

and for $m\in[u,u+1]$,

$$\begin{aligned}\mathbb{E}[v|m,\omega_0]&=\frac{\mathbb{P}_F[u](2\epsilon(m-u)+(1-\epsilon)\xi)}{\mathbb{P}_F[u](2\epsilon(m-u)+(1-\epsilon)\xi)+\mathbb{P}_F[0](u+1-m)\epsilon q^*/u}u\\ &=\frac{2\epsilon(m-u)+(1-\epsilon)\xi}{2\epsilon(m-u)+(1-\epsilon)\xi+\mathbb{P}_F[0](u+1-m)\epsilon\xi}u. \end{aligned}\tag{17}$$

In particular, $\mathbb{E}[v|u+1,\omega_0]=u$ and $\mathbb{E}[v|u,\omega_0]=(1-\epsilon)u/(1-\epsilon+\mathbb{P}_F[0]\epsilon)$, that is, $\mathbb{E}[v|m, \omega_0]$ is continuous in m at $m = u$.

We next consider the willingness to pay under signal structure $T_{IV}(\epsilon)$. Combining (15) with (16) and (17), the willingness to pay upon signal $m \in [q^*, u+1]$ is

$$\begin{aligned}
\hat{v}(m) &= \alpha \mathbb{E}[v|m, \omega_0] + (1-\alpha)\mathbb{E}[v|m, \omega \neq \omega_0] \\
&= \begin{cases} \left(1 - \alpha \dfrac{\mathbb{P}_F[0]\epsilon}{1-\epsilon+\mathbb{P}_F[0]\epsilon}\right) m & \text{if } m \in [q^*, u), \\ \left(1 - \alpha \dfrac{\mathbb{P}_F[0](u+1-m)\epsilon\xi}{2\epsilon(m-u)+(1-\epsilon)\xi+\mathbb{P}_F[0](u+1-m)\epsilon\xi}\right) u & \text{if } m \in [u, u+1] \end{cases} \\
&= \begin{cases} \kappa(u)m & \text{if } m \in [q^*, u), \\ \kappa(m)u & \text{if } m \in [u, u+1]. \end{cases}
\end{aligned}$$

Moreover, $\hat{v}(m_+) = \mathbb{E}[v|m_+, \omega_0]$ since m_+ is not in the support of H_v^{**} for any v. As $\hat{v}(m)$ is continuous and strictly increasing on $[q^*, u+1]$, we may define the inverse function $\hat{v}^{-1}(y) = \min\{m \in M : \hat{v}(m) = y\}$.

From the seller's perspective, the signal m is drawn from $\bar{H}^{**}$ on $[q^*, u+1]$. The willingness to pay $y = \hat{v}(m)$ is then distributed according to

$$\bar{H}^{**}(\hat{v}^{-1}(y)) = J^u_{\kappa(u)q^*}(y),$$

where we have used that $J^u_{q^*}(y/\kappa(u)) = J^u_{\kappa(u)q^*}(y)$ for $y < \hat{v}(u) = \kappa(u)u$. Consequently, the signal structure $T_{IV}(\epsilon)$ implements every price $p \in [\kappa(u)q^*, u]$ and revenue

$$R = \kappa(u)q^* = \left(1 - \alpha \frac{\mathbb{P}_F[0]\epsilon}{(1-\epsilon)+\mathbb{P}_F[0]\epsilon}\right) q^*.$$

Hence, each $R \in [(1-\alpha)q^*, q^*)$ is implemented for some $\epsilon \in (0, 1]$.

Finally, consider the ex ante utility. Fix $\epsilon \in (0, 1]$ and thus the implemented revenue $R = \kappa(u)q^*$. For any price $p \in [R, u]$, let $\tilde{U}_0(p)$ and $\tilde{U}_1(p)$ be the ex ante expected payoff in state ω_0 and $\omega \neq \omega_0$, respectively. That is,

$$\begin{aligned}
\tilde{U}_0(p) &= \int_M (\mathbb{E}[v|m, \omega_0] - p)\mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}^X(m), \\
\tilde{U}_1(p) &= \int_M (\mathbb{E}[v|m, \omega] - p)\mathbf{1}(\hat{v}(m) \geq p) \mathrm{d}\bar{H}^{**}(m),
\end{aligned}$$

where we used that, by the definition of conditional expectations (see Footnote 12),

$$\int_0^u \int_M (v-p)\mathbf{1}(\hat{v}(m) \geq p)\mathrm{d}H_{v,\omega}(m)\mathrm{d}F(v) = \int_M (\mathbb{E}[v|m,\omega]-p)\mathbf{1}(\hat{v}(m) \geq p)\mathrm{d}\bar{H}_\omega(m).$$

Note that $\tilde{U}_1(p)$ is continuous in p since $\bar{H}^{**}$ has support $[q^*, u+1]$ and is continuous for all $m \in [q^*, u+1)$, $\hat{v}$ is continuous and strictly increasing on $[q^*, u+1]$, and also $\mathbb{E}[v|m,\omega]$ is continuous in m on $[q^*, u+1]$. $\tilde{U}_0(p)$ can be written as

$$\tilde{U}_0(p) = \int_{\hat{v}^{-1}(p)}^{u+1} (\mathbb{E}[v|m,\omega_0] - p)\mathrm{d}\bar{H}^X(m) + \mathbb{P}_{\bar{H}^X}[m_+] \max\{\mathbb{E}[v|m_+,\omega_0] - p, 0\},$$

which is continuous in p since $\mathbb{E}[v|m,\omega_0]$ and $\bar{H}^X(m)$ are continuous in m on $[q^*, u+1]$. The ex ante utility implemented under price p is

$$\tilde{U}_2(p) = \alpha \min\{\tilde{U}_0(p), \tilde{U}_1(p)\} + (1-\alpha)\max\{\tilde{U}_0(p), \tilde{U}_1(p)\}.$$

$\tilde{U}_2(p)$ is continuous in p because $\tilde{U}_0(p)$ and $\tilde{U}_1(p)$ are continuous. Consider $p = u$. In this case, $\hat{v}(m) \geq p$ if and only if $m = u+1$. As $\mathbb{E}[v|u+1,\omega_0] = \mathbb{E}[v|u+1,\omega] = u$, we have $\tilde{U}_0(u) = \tilde{U}_1(u) = \tilde{U}_2(u) = 0$. Now, consider $p = \kappa(u)q^* = R$. Then $\hat{v}(m) \geq p$ for all $m \in [q^*, u+1]$, and $\tilde{U}_1(R) = \mathbb{E}_F[v] - R$. Moreover,

$$\begin{aligned}\tilde{U}_0(R) = &\int_{q^*}^{u+1} (\mathbb{E}[v|m,\omega_0] - R)\mathrm{d}\bar{H}^X(m) + \mathbb{P}_{\bar{H}^X}[m_+] \max\{\mathbb{E}[v|m_+,\omega_0] - R, 0\} \\ &\geq \mathbb{E}_F[v] - R,\end{aligned}$$

and thus $\tilde{U}_2(R) \geq \mathbb{E}_F[v] - R$. It follows that each ex ante utility $U = \tilde{U}_2(p) \in [0, \mathbb{E}_F[v] - R]$ is implemented for some price $p \in [R, u]$. □

Proof of Corollary 1. Corollary 1 follows directly from Theorem 2. □

Proof of Corollary 2. Corollary 2 follows directly from Theorem 2. □

Proof of Corollary 3. Part (i) is established in the main text. To prove part (ii), note that revenue R is implementable without relevant interim ambiguity if and only if is implementable by unambiguous signal structures, that is, $R \in [q^*, \mathbb{E}_F[v]]$ by Proposition 1(ii). Given any such R, Proposition 5(ii) establishes that only U within the bounds stated in part (ii) of the corollary are implementable. Under Assumption A1 with $\delta = 1$, the signal structures T_{III} defined for Region III in the proof of Theorem 2 implement all corresponding outcomes (U, R) without relevant interim ambiguity. □

Proof of Proposition 6(i). Let $\alpha \in (0,1]$. We start with some preliminaries. For $B \in [\mathbb{E}_F[v], u]$, let the CDF K_B with support $[q_B, B] \cup \{u\}$ be defined by

$$K_B(v) = \begin{cases} 1 - q_B/v & \text{if } v \in [q_B, B), \\ 1 - q_B/B & \text{if } v \in [B, u), \\ 1 & \text{if } v = u, \end{cases}$$

where $q_B \in (0, B)$ uniquely solves

$$q_B \ln(q_B) - q_B \ln(B) + u(1 - q_B/B) = \int_0^u F(v)\mathrm{d}v, \tag{18}$$

that is, q_B is such that $\int_0^u K_B(v)\mathrm{d}v = \int_0^u F(v)\mathrm{d}v$. Note that $K_u = J_{q^*}^u$ and that $K_B(v)$ is continuous in B for all $v \in [0, u]$. Hence, as Assumption A2 holds with strict inequality in (5) for all $x \in (0, u)$, there is $B' < u$ such that for all $B \in [B', u]$,

$$\int_0^x F(v)\mathrm{d}v \geq \int_0^x K_B(v)\mathrm{d}v \quad \text{for all } x \in [0, u],$$

that is, F is a mean-preserving spread of K_B.

Moreover, define $Z(\psi_0, \psi_u) = \psi_u \mathbb{P}_F[u]u/(\psi_0 \mathbb{P}_F[0] + \psi_u \mathbb{P}_F[u])$. Let

$$\underline{z} = \min_{(\psi_0, \psi_u) \in [0,1]^2} Z(\psi_0, \psi_u) \ \text{ s.t. } \ \psi_0 \mathbb{P}_F[0] + \psi_u \mathbb{P}_F[u] = \mathbb{P}_F[u]$$

and $B'' = \alpha \max\{\mathbb{E}_F[v], \underline{z}\} + (1 - \alpha)u$.

From now on, we fix $B \in [\max\{B', B''\}, u)$ and let q_B be as defined in (18), where $q_B < q^*$ since $B < u$. Moreover, we fix $(\psi_0, \psi_u) \in [0,1]^2$ such that

$$\alpha Z(\psi_0, \psi_u) + (1 - \alpha)u = B \quad \text{and} \quad \psi_0 \mathbb{P}_F[0] + \psi_u \mathbb{P}_F[u] = \frac{q_B}{B}. \tag{19}$$

Note that such ψ_0, ψ_u exist since $B \geq B''$ and $q_B/B = \mathbb{P}_{K_B}[u] \leq \mathbb{P}_F[u]$ (as F is a mean-preserving spread of K_B).

We are ready to define a signal structure that consistently implements revenue $R = q_B < q^*$. Consider the signal structure H with signals $M = [q_B, B] \cup \{m_+\}$ for $m_+ > u$, distributions

$$H_{v,\omega_0}(v) = \begin{cases} (1 - \psi_0)\mathbf{1}(m \geq m_+) + \psi_0 \mathbf{1}(m \geq B) & \text{if } v = 0, \\ \mathbf{1}(m \geq m_+) & \text{if } v \in (0, u), \\ (1 - \psi_u)\mathbf{1}(m \geq m_+) + \psi_u \mathbf{1}(m \geq B) & \text{if } v = u, \end{cases}$$

and for all $\omega \neq \omega_0$, distributions $H_{v,\omega}$ such that $\bar{H}_\omega(m) = J^B_{q_B}(m)$ with $\mathbb{E}[v|m,\omega] = m$ for $m < B$ and $\mathbb{E}[v|m,\omega] = u$ for $m = B$. Such $H_{v,\omega}$ exist as $\mathbb{E}[v|m,\omega]$ is distributed according to K_B and F is a mean-preserving spread of K_B.

Observe that by (19), $\mathbb{E}[v|B,\omega_0] = Z(\psi_0,\psi_u) \in [\mathbb{E}_F[v], u)$ and $\mathbb{P}_{\bar{H}_{\omega_0}}[B] = q_B/B = \mathbb{P}_{\bar{H}_\omega}[B]$ for $\omega \neq \omega_0$. Because $\int_M \mathbb{E}[v|m,\omega_0]\mathrm{d}\bar{H}_{\omega_0}(m) = \int_M \mathbb{E}[v|m,\omega]\mathrm{d}\bar{H}_\omega(m)$ $= \mathbb{E}_F[v]$ for $\omega \neq \omega_0$, we have

$$\left(1 - \frac{q_B}{B}\right)\mathbb{E}[v|m_+,\omega_0] + \frac{q_B}{B}Z(\psi_0,\psi_u) = \left(1 - \frac{q_B}{B}\right)\mathbb{E}_{K_B}[v|v < B] + \frac{q_B}{B}u = \mathbb{E}_F[v],$$

which implies $\mathbb{E}[v|m_+,\omega_0] > \mathbb{E}_{K_B}[v|v < B] > q_B$ and $\mathbb{E}[v|m_+,\omega_0] \leq \mathbb{E}_F[v]$. Any signal $m \neq B$ perfectly reveals the payoff-irrelevant state. The willingness to pay is

$$\hat{v}(m) = \begin{cases} m & \text{if } m \in [q_B, B), \\ \alpha Z(\psi_0,\psi_u) + (1-\alpha)u = B & \text{if } m = B, \\ \mathbb{E}[v|m_+,\omega_0] \in (q_B, \mathbb{E}_F[v]] & \text{if } m = m_+, \end{cases}$$

where $\hat{v}(B) = B$ because of (19).

We next show that $a^*_p(m) = \mathbf{1}(\hat{v}(m) \geq p)$ maximizes $U(a,p)$. Using (7),

$$\begin{aligned} U_G(a,p) &= \int_\Omega \int_M (\mathbb{E}[v|m,\omega] - p)a(m)\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G(\omega) \\ &= \mathbb{P}_G[\omega_0]\left(\frac{q_B}{B}(Z(\psi_0,\psi_u) - p)a(B) + \frac{B - q_B}{B}(\mathbb{E}[v|m_+,\omega_0] - p)a(m_+)\right) \\ &\quad + (1 - \mathbb{P}_G[\omega_0])\left(\frac{q_B}{B}(u - p)a(B) + \int_{[q_B,B)} (m - p)a(m)\mathrm{d}J^B_{q_B}(m)\right). \end{aligned}$$

Clearly, $a(m) = a^*_p(m)$ for all $m \neq B$ maximizes $U_G(a,p)$ for all G and thus $U(a,p)$, and we may restrict attention to such a. Moreover, for $p \notin (\mathbb{E}_F[v], u)$, $a(B) = a^*_p(B)$ maximizes $U_G(a,p)$ for all G. For $p \in (\mathbb{E}_F[v], u)$ and a such that $a(m) = a^*_p(m)$ for $m \neq B$, we have

$$\begin{aligned} U_G(a,p) =& \frac{q_B}{B}[\mathbb{P}_G[\omega_0]Z(\psi_0,\psi_u) + (1 - \mathbb{P}_G[\omega_0])u - p]a(B) \\ &+ (1 - \mathbb{P}_G[\omega_0])\int_{[q_B,B)} \max\{m - p, 0\}\mathrm{d}J^B_{q_B}(m). \end{aligned}$$

As $\mathbb{P}_G[\omega_0] = 1$ and $\mathbb{P}_G[\omega_0] = 0$, respectively, minimizes and maximizes $U_G(a,p)$,

$$U(a,p) = \frac{q_B}{B}(\hat{v}(B) - p)a(B) + (1-\alpha)\int_{[q_B,B)} \max\{m - p, 0\}\mathrm{d}J^B_{q_B}(m).$$

Hence, $a(B) = a^*_p(B)$ maximizes $U(a,p)$, that is, H induces dynamically consistent behavior.

From the seller's perspective, $\hat{v}(m)$ is drawn from $J^B_{q_B}$, so that H consistently implements price q_B, revenue $q_B < q^*$, and ex ante utility $U(a^*_{q_B}, q_B) = \mathbb{E}_F[v] - q_B > \mathbb{E}_F[v] - q^*$. □

Proof of Proposition 6(ii.a). Assumption A2 implies $B^* = u$. Let $\theta \in (0,1]$ and consider the signal structure H with signals $M = [q^*, u] \cup [2u, 3u]$ and distributions

$$H_{v,\omega}(m) = \begin{cases} \theta J^u_{q^*}(m) + (1-\theta)\mathbf{1}(m \geq 2u) & \text{if } \omega = \omega_0 \text{ and } v = 0, \\ \mathbf{1}(m \geq 2u + v) & \text{if } \omega = \omega_0 \text{ and } v > 0, \\ H^*_v(m) & \text{if } \omega \neq \omega_0. \end{cases}$$

For all $\omega \neq \omega_0$, H coincides with the unambiguous benchmark H^*, so that $\bar{H}_\omega = J^u_{q^*}$ and $\mathbb{E}[v|m,\omega] = m$ for all $m \in [q^*, u]$. In state ω_0, $\bar{H}_{\omega_0}(m) = \theta F(0) J^u_{q^*}(m) + F(m-2u) - \theta F(0)\mathbf{1}(m \geq 2u)$ and $\mathbb{E}[v|m,\omega_0] = 0$ for all $m \in [q^*, u]$, whereas signals $m \in [2u, 3u]$ unambiguously reveal the valuation to be $v = m - 2u$. Accordingly, $\hat{v}(m) = (1-\alpha)m$ for $m \in [q^*, u]$ and $\hat{v}(m) = m - 2u$ for $m \in [2u, 3u]$. It is routine to check that H implements revenue $R = (1-\alpha)q^*$ as m is drawn from $J^u_{q^*}$ under the seller's prior G_S. In the remainder of the proof we will show that there is $\alpha' > 0$ such that for all $\alpha \in [0, \alpha']$, $U(a^*_p, p) = \max_a U(a,p)$ for all p, that is, H consistently implements $R = (1-\alpha)q^*$.

As there is no relevant interim ambiguity after signals $m \in [2u, 3u]$, we restrict attention to purchase plans a such that $a(m) = a^*_p(m)$ for $m \in [2u, 3u]$. Using (7),

$$\begin{aligned} U_G(a,p) &= \int_\Omega \int_M (\mathbb{E}[v|m,\omega] - p) a(m) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G(\omega) \\ &= \mathbb{P}_{G_0}[\omega_0] \left(\mathbb{E}_F[\max\{v-p, 0\}] - \int_{q^*}^u \theta F(0) p a(m) \mathrm{d}J^u_{q^*}(m) \right) \\ &\quad + (1 - \mathbb{P}_{G_0}[\omega_0]) \int_{q^*}^u (m-p) a(m) \mathrm{d}J^u_{q^*}(m). \end{aligned}$$

Let G_0 be such that $\mathbb{P}_{G_0}[\omega_0] = 1$ and let G_1 be such that $\mathbb{P}_{G_1}[\omega_0] = 0$. Moreover, let $\phi^I = 1 + \theta F(0)(1-\alpha)/\alpha$ and $\phi^{II} = 1 + \theta F(0)\alpha/(1-\alpha)$, and define

$$\begin{aligned} U^I(a,p) &= \alpha U_{G_1}(a,p) + (1-\alpha)U_{G_0}(a,p) \\ &= \int_{q^*}^{u} \alpha(m - \phi^I p)a(m)\mathrm{d}J^u_{q^*}(m) + (1-\alpha)\mathbb{E}_F[\max\{v-p,0\}], \qquad (20) \\ U^{II}(a,p) &= \alpha U_{G_0}(a,p) + (1-\alpha)U_{G_1}(a,p) \\ &= \int_{q^*}^{u} (1-\alpha)(m - \phi^{II} p)a(m)\mathrm{d}J^u_{q^*}(m) + \alpha\mathbb{E}_F[\max\{v-p,0\}]. \qquad (21) \end{aligned}$$

Note that $U(a,p) \in \{U^I(a,p), U^{II}(a,p)\}$ because U_G is minimized and maximized, respectively, either by G_1 and G_0 or by G_0 and G_1.

From now on, we assume $\alpha \le \sqrt{F(0)}/(1+\sqrt{F(0)})$, which implies $\alpha < 1/2$, and we let

$$\theta = \frac{\alpha^2}{(1-\alpha)^2 F(0)}.$$

Then, $U(a,p) = \max\{U^I(a,p), U^{II}(a,p)\}$ because $\alpha < 1/2$ implies

$$\alpha \min_G U_G(a,p) + (1-\alpha)\max_G U_G(a,p) \ge \alpha \max_G U_G(a,p) + (1-\alpha)\min_G U_G(a,p).$$

Moreover, $\max_a U(a,p) = \max\{\max_a U^I(a,p), \max_a U^{II}(a,p)\}$. By (20), the purchase plan a^I such that $a^I(m) = \mathbf{1}(m/\phi^I \ge p)$ for $m \in [q^*, u]$ maximizes $U^I(a,p)$. Similarly, by (21), a^{II} such that $a^{II}(m) = \mathbf{1}(m/\phi^{II} \ge p)$ for $m \in [q^*, u]$ maximizes $U^{II}(a,p)$. Now, note that $1/\phi^I = 1-\alpha$ and thus $a^I = a^*_p$. Consequently, $U(a^*_p, p) = U^I(a^I, p)$, and we are left to show that $U^I(a^I,p) \ge U^{II}(a^{II},p)$ for all p.

As $\alpha < 1/2$, $\phi^I \ge \phi^{II}$ (where equality holds for $\alpha = 0$). Hence, $a^I = a^{II}$ and thus $U^I(a^I,p) = U^{II}(a^{II},p)$ if $p \le q^*/\phi^I$ and if $p > u/\phi^{II}$, so that we are left to consider $q^*/\phi^I < p \le u/\phi^{II}$. We have

$$\begin{aligned} U^I(a^I,p) &= \alpha \int_{q^*}^{u} \max\{m - \phi^I p, 0\}\mathrm{d}J^u_{q^*}(m) + (1-\alpha)\mathbb{E}_F[\max\{v-p,0\}], \\ U^{II}(a^{II},p) &= (1-\alpha) \int_{q^*}^{u} \max\{m - \phi^{II} p, 0\}\mathrm{d}J^u_{q^*}(m) + \alpha\mathbb{E}_F[\max\{v-p,0\}]. \end{aligned}$$

We separately consider two cases. For each case, we show that there is $\alpha' > 0$ such that $U^I(a^I,p) - U^{II}(a^{II},p) \ge 0$ for all $\alpha \le \alpha'$.

The first case is $p \in (q^*/\phi^I, u/\phi^I)$. Note that for such prices p, there is $s \in (0,1)$

such that $p = p_s(\alpha) = [(1-s)q^* + su]/\phi^I$. At price $p_s(\alpha)$ for any fixed s, let

$$\begin{aligned}\Delta(\alpha) &= U^I(a^I, p_s(\alpha)) - U^{II}(a^{II}, p_s(\alpha)) \\ &= (1-2\alpha)\mathbb{E}_F[\max\{v - p_s(\alpha), 0\}] + \alpha \int_{q^*}^{u} \max\{m - \phi^I p_s(\alpha), 0\}\mathrm{d}J_{q^*}^u(m) \\ &\quad - (1-\alpha)\int_{q^*}^{u} \max\{m - \phi^{II} p_s(\alpha), 0\}\mathrm{d}J_{q^*}^u(m).\end{aligned}$$

Note that $\Delta(\alpha)$ is continuous in α. Moreover

$$\begin{aligned}\Delta(0) &= \int_0^u \max\{v - p_s(0), 0\}\mathrm{d}F(v) - \int_{q^*}^{u} \max\{m - p_s(0), 0\}\mathrm{d}J_{q^*}^u(m) \\ &= \int_{p_s(0)}^{u} [1 - F(v)]\mathrm{d}v - \int_{p_s(0)}^{u} [1 - J_{q^*}^u(m)]\mathrm{d}m \\ &= \int_{p_s(0)}^{u} J_{q^*}^u(m)\mathrm{d}m - \int_{p_s(0)}^{u} F(v)\mathrm{d}v \\ &> 0,\end{aligned}$$

where the second equality uses integration by parts and the inequality uses $p_s(0) \in (q^*, u)$ and that Assumptions A2 holds with strict inequality in (5) for all $x \in (0, u)$. Hence, there is $\alpha' > 0$ such that $\Delta(\alpha) \geq 0$ for $\alpha \leq \alpha'$.

The second case is $p \in [u/\phi^I, u/\phi^{II}]$. For any such p, $U^I(a^I, p) - U^{II}(a^{II}, p) \geq \underline{\Delta}(p, \alpha)$, where

$$\underline{\Delta}(p, \alpha) = (1-2\alpha)\mathbb{P}_F[u](u - p) - (1-\alpha)\int_{\phi^{II}p}^{u} (m - \phi^{II} p)\mathrm{d}J_{q^*}^u(m).$$

Note that $\underline{\Delta}(p, \alpha)$ is a concave function of p, and $\underline{\Delta}(u/\phi^{II}, \alpha) \geq 0$. Moreover,

$$\underline{\Delta}(u/\phi^I, \alpha) = (1-2\alpha)\mathbb{P}_F[u]u\alpha - (1-\alpha)\int_{uz(\alpha)}^{u} [m - uz(\alpha)]\mathrm{d}J_{q^*}^u(m),$$

where $z(\alpha) = \phi^{II}/\phi^I$. Taking the derivative with respect to α yields

$$\frac{\partial \underline{\Delta}(u/\phi_I, \alpha)}{\partial \alpha} = (1-4\alpha)\mathbb{P}_F[u]u + \int_{uz(\alpha)}^{u} [m - uz(\alpha)]\mathrm{d}J_{q^*}^u(m) + (1-\alpha)\int_{uz(\alpha)}^{u} uz'(\alpha)\mathrm{d}J_{q^*}^u(m).$$

Now, observe that $\lim_{\alpha\to 0} \underline{\Delta}(u/\phi^I, \alpha) = \underline{\Delta}(u, 0) = 0$ and

$$\left.\frac{\partial \underline{\Delta}(u/\phi^I, \alpha)}{\partial \alpha}\right|_{\alpha=0} = \mathbb{P}_F[u]u + \int_{u}^{u} uz'(0)\mathrm{d}J_{q^*}^u(m) = \mathbb{P}_F[u]u - q^* > 0,$$

where the second equality uses $z'(0) = -1$ and the inequality is implied by Assumption A2 holding with strict inequality in (5) for all $x \in (0, u)$. Consequently, there is $\alpha' > 0$ such that $\underline{\Delta}(u/\phi^I, \alpha) \geq 0$ for all $\alpha \leq \alpha'$. In turn, for such α, $U^I(a^I, p) - U^{II}(a^{II}, p) \geq \underline{\Delta}(p, \alpha) \geq 0$ for all $p \in [u/\phi^I, u/\phi^{II}]$. □

Proof of Proposition 6(ii.b). Consider a signal structure H that implements revenue $(1-\alpha)q^*$. Hence, from the seller's perspective

$$p \int_\Omega \int_M a_p^*(m) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G_S(\omega) \leq (1-\alpha)q^* \quad \text{for all } p. \tag{22}$$

Now, fix a price $p \in (q^*, \mathbb{E}_F[v])$, which exists as $q^* < \mathbb{E}_F[v]$. Note that

$$\begin{aligned}
\min_G U_G(a_p^*, p) \leq U_{G_S}(a_p^*, p) &= \int_\Omega \int_0^u \int_M (v-p) a_p^*(m) \mathrm{d}H_{v,\omega}(m) \mathrm{d}F(v) \mathrm{d}G_S(\omega) \\
&\leq (u-p) \int_\Omega \int_M a_p^*(m) \mathrm{d}\bar{H}_\omega(m) \mathrm{d}G_S(\omega) \\
&\leq \frac{u-p}{p}(1-\alpha)q^*,
\end{aligned}$$

where the last line uses (22). Moreover, as $\max_G U_G(a_p^*, p) \leq \mathbb{E}_F[\max\{v-p, 0\}]$,

$$U(a_p^*, p) \leq \alpha \frac{u-p}{p}(1-\alpha)q^* + (1-\alpha)\mathbb{E}_F[\max\{v-p, 0\}].$$

Note that the right-hand side is continuous in α, equals $\mathbb{E}_F[\max\{v-p, 0\}]$ for $\alpha = 0$, and equals 0 for $\alpha = 1$. Hence, there is $0 < \alpha'' < 1$ such that

$$U(a_p^*, p) < \mathbb{E}_F[v] - p \quad \text{for all } \alpha \in (\alpha'', 1].$$

The purchase plan $a(m) = 1$ for all $m \in M$ yields $U(a, p) = \mathbb{E}_F[v] - p$. Hence, for $\alpha \in (\alpha'', 1]$, $U(a_p^*, p) < U(a, p)$, that is, H fails to induce dynamically consistent behavior at p. □

Proof of Proposition 7(i). We prove that revenue $R > \mathbb{E}_F[v]$ is consistently implementable by showing that signal structure SE in Subsection 5.3 induces dynamically consistent behavior, generalizing the analysis in the main text to arbitrary $\delta \in (0, 1]$. As SE induces no relevant interim ambiguity after all $m \neq m_3$, we can focus on plans a_0 and a_1 with $a_0(m_3) = 0$, $a_1(m_3) = 1$, and $a_0(m) = a_1(m) = a_p^*(m)$ for $m \in \{m_1, m_2\}$ when maximizing $U(a, p)$. As in the main text, a_p^* clearly maximizes $U(a, p)$ if $p \leq \mathbb{E}_F[v] - \sigma\epsilon$. For $p > \mathbb{E}_F[v] - \sigma\epsilon$ and $a \in \{a_0, a_1\}$, $U_G(a, p)$ is as in (9). Note that $\mathbb{P}_G[\omega_0] = \delta$ minimizes $U_G(a_0, p)$ whereas $\mathbb{P}_G[\omega_0] = 0$

minimizes $U_G(a_1, p)$. Hence, the plans a_0 and a_1 yield

$$\begin{aligned}
U(a_0, p) &= (1-\alpha\delta)\max\{\mathbb{E}_F[v]+\sigma\epsilon-p, 0\}/2, \\
U(a_1, p) &= (1-\alpha)\delta(\mathbb{E}_F[v]+\epsilon-p)/2 \\
&\quad + [1-(1-\alpha)\delta][\max\{\mathbb{E}_F[v]+\sigma\epsilon-p, 0\}+(\mathbb{E}_F[v]-\sigma\epsilon-p)]/2 \\
&= [1-(1-\alpha)\delta]\max\{\mathbb{E}_F[v]+\sigma\epsilon-p, 0\}/2+(\mathbb{E}_F[v]+\sigma\epsilon-p)/2.
\end{aligned}$$

Since $U(a_1, p) \geq U(a_0, p)$ if and only if $p \leq \mathbb{E}_F[v]+\sigma\epsilon = \hat{v}(m_3)$, $a = a_p^*$ maximizes $U(a, p)$ for all p, that is, SE induces dynamically consistent behavior. □

Proof of Proposition 7(ii). Consider any outcome (U, R) with $R \in (\mathbb{E}_F[v], R^*]$ and $U \in [0, \bar{U}(R)]$. Since the prior F has support $\{0, u\}$, $\mathbb{E}_F[v] = [1-F(0)]u$,

$$\bar{U}(R) = \alpha(\mathbb{E}_F[v]-R)+(1-\alpha)\mathbb{E}_F[\max\{v-R, 0\}] = \mathbb{E}_F[v]-[1-(1-\alpha)F(0)]R,$$

and $R^* = \bar{U}^{-1}(0) = \mathbb{E}_F[v]/[1-(1-\alpha)F(0)]$ as defined in Corollary 2. We next define a signal structure H with $M = \{m_1, m_2, m_3, m_4\}$ that consistently implements (U, R). In state ω_0, H draws signals m_1, m_2, and m_3 such that

$$\mathbb{P}_{\bar{H}_{\omega_0}}[m] = \begin{cases} \rho_1 & \text{if } m = m_1, \\ 1-\rho_1-\rho_4 & \text{if } m = m_3, \\ \rho_4 & \text{if } m = m_4 \end{cases} \quad \text{and} \quad \mathbb{E}[v|m, \omega_0] = \begin{cases} 0 & \text{if } m = m_1, \\ \overline{E} & \text{if } m = m_3, \\ u & \text{if } m = m_4, \end{cases}$$

whereas in any state $\omega \neq \omega_0$, H draws signals m_2 and m_3 such that

$$\mathbb{P}_{\bar{H}_{\omega}}[m] = \begin{cases} \rho_1+\rho_4 & \text{if } m = m_2, \\ 1-\rho_1-\rho_4 & \text{if } m = m_3 \end{cases} \quad \text{and} \quad \mathbb{E}[v|m, \omega] = \begin{cases} R & \text{if } m = m_2, \\ \underline{E} & \text{if } m = m_3, \end{cases}$$

where $\rho_1, \rho_4, \underline{E}, \overline{E}$ are the unique solution to the system of linear equations

$$(1-\rho_1-\rho_4)\overline{E}+\rho_4 u = \mathbb{E}_F[v], \tag{23}$$

$$(1-\rho_1-\rho_4)\underline{E}+(\rho_1+\rho_4)R = \mathbb{E}_F[v], \tag{24}$$

$$\alpha\underline{E}+(1-\alpha)\overline{E} = R, \tag{25}$$

$$\mathbb{E}_F[v]-[1-(1-\alpha)\rho_1]R = U. \tag{26}$$

Equations (23) and (24) ensure that there are distributions $H_{v,\omega}$ that are consistent with $\bar{H}_\omega$ and $\mathbb{E}[v|m,\omega]$. As is straightforward to verify, the solution satisfies

$$\rho_1 = \frac{R - \mathbb{E}_F[v] + U}{(1-\alpha)R} \in [0, F(0)], \qquad \underline{E} = R\frac{[(2-\alpha)\mathbb{E}_F[v] - R](u-R) - Uu}{(\mathbb{E}_F[v] - \alpha R)(u-R) - Uu} \le R,$$
$$\rho_4 = \frac{U}{(1-\alpha)(u-R)} \in [0, 1-F(0)], \quad \overline{E} = R\frac{(1-\alpha)\mathbb{E}_F[v](u-R) - Uu}{(\mathbb{E}_F[v] - \alpha R)(u-R) - Uu} \in [R, u].$$

Note that the denominators on the right-hand side are positive since

$$Uu \le \bar{U}(R)u = \mathbb{E}_F[v](u-R) - \alpha R(u - \mathbb{E}_F[v]). \tag{27}$$

Moreover, the assumptions in parts (ii.a) and (ii.b) of the proposition imply $\underline{E} \ge 0$: Using (27), the numerator in the solution for $\underline{E}$ satisfies

$$[(2-\alpha)\mathbb{E}_F[v] - R](u-R) - Uu \ge [R - (1-\alpha)u](R - \mathbb{E}_F[v]),$$

so that $\underline{E} \ge 0$ if $R \ge (1-\alpha)u$. Now, $R^* \ge (1-\alpha)u$ if $\alpha \ge 1 - [1-F(0)]/F(0)$ as in part (ii.a), which focuses on revenue R^*, whereas $R \ge \mathbb{E}_F[v] \ge (1-\alpha)u$ holds for all R if $\alpha \ge F(0)$ as in part (ii.b).

The signal structure induces relevant interim ambiguity only after signal m_3, where $\hat{v}(m_3) = R$ by (25). For the other signals, $\hat{v}(m_1) = 0$, $\hat{v}(m_2) = R$, and $\hat{v}(m_4) = u$. From the seller's perspective, $\hat{v}(m) = R$ with probability one. Hence, H implements price and revenue R. Given price R, the buyer ex ante expects to always buy in the worst case and to avoid buying upon learning that $v = 0$ with probability ρ_1 in the best case. Hence, H implements ex ante utility U by (26).

We are left to show that the buyer's behavior is dynamically consistent at any price $p \in [0, u]$. As there is no relevant interim ambiguity after $m \in \{m_1, m_2, m_4\}$, $a(m) = a_p^*(m) = \mathbf{1}(\hat{v}(m) \ge p)$ for those m maximizes $U_G(a, p)$. For such a plan a,

$$\begin{aligned} U_G(a,p) &= \int_\Omega \int_M (\mathbb{E}[v|m,\omega] - p)a(m)\mathrm{d}\bar{H}_\omega(m)\mathrm{d}G(\omega) \\ &= \mathbb{P}_G[\omega_0][(1-\rho_1-\rho_4)(\overline{E} - p)a(m_3) + \rho_4(u-p)] \\ &\quad + (1 - \mathbb{P}_G[\omega_0])[(1-\rho_1-\rho_4)(\underline{E} - p)a(m_3) + (\rho_1+\rho_4)\max\{R-p, 0\}]. \end{aligned}$$

Recall $\underline{E} \le \overline{E}$. Clearly, for $p \notin (\underline{E}, \overline{E})$, $a(m_3) = a_p^*(m_3)$ maximizes U_G for all G.

We next consider $p \in [\tilde{p}, \overline{E})$ with $\tilde{p} = \max\{R - (u-R)\rho_4/\rho_1, \underline{E}\}$. Note that $p \ge \tilde{p}$ implies $\rho_4(u-p) \ge (\rho_1+\rho_4)(R-p)$. Hence, in this case, $\mathbb{P}_G[\omega_0] = 0$

minimizes U_G. Consequently,

$$\begin{aligned}U(a,p) &= (1-\alpha)[(1-\rho_1-\rho_4)(\overline{E}-p)a(m_3)+\rho_4(u-p)] \\ &\quad+\alpha[(1-\rho_1-\rho_4)(\underline{E}-p)a(m_3)+(\rho_1+\rho_4)\max\{R-p,0\}] \\ &= (1-\rho_1-\rho_4)(\hat{v}(m_3)-p)a(m_3)+(1-\alpha)\rho_4(u-p)+\alpha(\rho_1+\rho_4)\max\{R-p,0\},\end{aligned}$$

that is, $a(m_3) = a_p^*(m_3)$ maximizes U.

It remains to consider $p \in (\underline{E}, \tilde{p})$. For the plan a_1 with $a_1(m_3) = 1$ and $a_1(m) = a_p^*(m)$ otherwise,

$$\begin{aligned}U_G(a_1,p) &= \mathbb{P}_G[\omega_0][(1-\rho_1-\rho_4)(\overline{E}-p)+\rho_4(u-p)] \\ &\quad+(1-\mathbb{P}_G[\omega_0])[(1-\rho_1-\rho_4)(\underline{E}-p)+(\rho_1+\rho_4)(R-p)] \\ &= \mathbb{P}_G[\omega_0][\mathbb{E}_F[v]-(1-\rho_1)p]+(1-\mathbb{P}_G[\omega_0])[\mathbb{E}_F[v]-p],\end{aligned}$$

using (23) and (24) for the second equality. Hence, $\mathbb{P}_G[\omega_0] = 0$ minimizes $U_G(a_1, p)$. For the plan a_0 with $a_0(m_3) = 0$ and $a_0(m) = a_p^*(m)$ otherwise, $U_G(a_0, p) = \mathbb{P}_G[\omega_0]\rho_4(u-p)+(1-\mathbb{P}_G[\omega_0])(\rho_1+\rho_4)(R-p)$, which implies $\mathbb{P}_G[\omega_0] = 0$ maximizes $U_G(a_0, p)$ for $p \leq \tilde{p}$. Hence,

$$\begin{aligned}U(a_1,p) &= \mathbb{E}_F[v]-(1-(1-\alpha)\rho_1)p, \\ U(a_0,p) &= \alpha\rho_4(u-p)+(1-\alpha)(\rho_1+\rho_4)(R-p)\end{aligned}$$

for all $p \in [\underline{E}, \tilde{p}]$. We argued above that for $p \in \{\underline{E}, \tilde{p}\}$, $a(m_3) = a_p^*(m_3) = 1$ maximizes $U(a, p)$, that is, $U(a_1, p) - U(a_0, p) \geq 0$. As $U(a_1, p) - U(a_0, p)$ is an affine function of p, it follows that $U(a_1, p) - U(a_0, p) \geq 0$ for all $p \in [\underline{E}, \tilde{p}]$. Consequently, $a = a_p^*$ maximizes $U(a, p)$ for all p. □

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