A Continuum Model as a Limit of Large Finite Matching Markets

Aram Grigoryan

University of California, San Diego

Abstract

The continuum matching model has been instrumental in the studies of finite economies. Yet, there is limited theoretical justification on whether and when a continuum model approximates large finite problems. In this paper I study the following question: if we randomly sample finite economies from some distribution, will the stable assignments of these finite economies converge to a stable assignment of the continuum economy as we increase the sample size? I provide a simple condition, which I call rich preferences, that guarantees the convergence. I also provide approximate convergence results under weaker conditions.

1 Introduction

The matching model of Gale and Shapley (1962) is widely applied for studying real-life assignment problems such as residents-hospitals matching (Roth, 1984), college admissions (Balinski and Sönmez, 1999), and school choice (Abdulkadiroğlu and Sönmez, 2003). In this model, there are two sides of market participants, e.g. applicants and schools. Each market participant has a strict ranking over the participants on the opposite side of the market. For example, in school choice an applicant's ranking over the schools corresponds to her choice list submitted to the application system, and a school's ranking over the applicants is determined by the applicants' priority scores. These priorities, in turn, may depend on their exam scores, residential locations, and/or random lottery numbers. Gale and Shapley (1962) formulate the celebrated applicant-proposing deferred acceptance (DA) algorithm for finding a stable assignment in this problem. The continuum market (re)formulation of the Gale and Shapley (1962) model has been another important contribution for tractable analysis of matching markets. Azevedo and Leshno (2016) (hereafter, AL) and Abdulkadiroğlu, Che, and Yasuda (2015) pioneered the study of a model with a continuum of applicants and a finite number of schools. The continuum economy can be concisely described with a joint distribution over applicants' preference rankings and priority scores. Due to its tractability, the continuum matching model has been extensively used in theoretical and empirical literature on market design.

Since real-life markets feature only a finite set of applicants, one may be interested in whether the conclusions derived from the continuum model would also hold approximately in large finite markets that are sampled from the continuum market distribution. In this paper, I study the following question: if we draw random samples of applicants (and their preferences and priority scores) according to some distribution, will the DA outcome of these finite markets converge to the DA outcome of the continuum economy described with this distribution as the sample size increases? I refer to this as the DA convergence question.

There is limited theoretical justification for the DA convergence. The existing sufficient conditions for convergence are technical and/or potentially restrictive for several important applications. I provide a condition, which I call rich preferences, that guarantees the DA convergence. In a nutshell, the condition says that certain relative rankings of the outside option in the applicants' preferences are sufficiently represented in a sense that the sampling distribution puts some positive mass on them. The mass can be arbitrarily small. The condition is simple, testable, and potentially realistic in real-life applications.

Below I summarize the previous literature that provides sufficient conditions for the DA convergence.

In their pioneering work, AL provide sufficient conditions for the DA convergence in the continuum model. Their Proposition 3 says that the DA convergence is guaranteed whenever (i) the total capacity at all schools is smaller than the mass of applicants, and (ii) there is a unique stable assignment. In their Theorem 1, AL also give two separate sufficient conditions for having a unique stable assignment. The first condition assumes *full support*, which says that all combination of preferences and priorities profiles are represented in the economy. The second set of conditions, which guarantee uniqueness for a 'generic' problem, require that the 'demand functions',[1] are differentiable, and also that the total capacity at all schools is smaller than the mass of applicants.

If we combine Proposition 3 and Theorem 1 of AL, we get that the (generic) DA convergence is guaranteed whenever (i) the total capacity at all schools is smaller than the mass of applicants, and (ii) the economy has full support, or the the demand function is differentiable. The capacity restriction of AL excludes the possibility of

[1]The notion of demand functions will be introduced in Section 3.1.

having an *outside option* or an undersubscribed school. In many applications, including school choice or college admissions, there is always an outside option (equivalently, an option of not ranking certain schools). Thus, the capacity restriction of AL is violated *by design*. Additionally, in school choice, priorities are oftentimes (partially) determined by residential locations. As a result, the full support assumption will also be violated by design. For example, if an applicant has a neighborhood priority at one school, then the same applicant cannot have a neighborhood priority at a school that is outside of her neighborhood boundaries. Thus, there cannot be students that simultaneously have highest priority scores at every school. Finally, AL's differentiability of the demand condition is a technical one. It does not directly speak to the model's primitives (preferences and priority scores), and it guarantees convergence for a 'generic', but not all problems.[2]

Abdulkadiroğlu, Angrist, Narita, and Pathak (2017) also study the convergence problem. This influential paper introduces an important application for the continuum model, namely, a tractable estimation of school assignment probabilities for propensity score analysis. This approach builds on a DA convergence result. Potentially noticing that the AL conditions are violated in the school choice setup, Abdulkadiroğlu et al. (2017) suggest a weaker condition, called the *rich support*. However, the rich support assumption is not sufficient for the DA convergence. In Section 4, I provide a (counter)example to demonstrate this fact.

To summarize, the existing convergence results use assumptions that may be restrictive and/or technical. My conditions do not put restrictions on applicants' priority

[2]Two subsequent papers have revisited the problem and provided convergence results without the capacity restrictions of AL. Agarwal and Somaini (2018) obtain convergence without the capacity restrictions, but they maintain the differentiability assumption and impose further technical and abstract substitutability conditions on the demand function. In a recent paper, Artemov, Che, and He (2023) obtain convergence without the capacity restriction, but they maintain the full support assumption.

scores and capacities, and they only put arguable 'mild' restrictions on the preferences.

In addition to establishing the exact DA converge, I provide 'approximate' convergence results for the case when the rich preferences condition fails. I use a well-known result by Vapnik and Chervonenkis (1971) to establish the approximate and exact DA convergence. I also show that under rich preferences there is a unique stable assignment.

My convergence and uniqueness results strengthen the existing theoretical foundation for the the continuum model analysis of large finite matching markets.

2 Related Literature

The continuum matching model has been widely used in both theoretical and empirical literature on market design. Azevedo and Leshno (2016) highlight the importance of their continuum model for tractable analysis of schools' incentives to improve quality. Since then, it has been a workhorse model for tractable analyses of various questions in school choice and college admissions, such as applicants' welfare (Bodoh-Creed, 2020; Leshno and Lo, 2021), segregation (Escobar and Huerta, 2021), priority design (Celebi and Flynn, 2021), efficient seat reassignment (Feigenbaum, Kanoria, Lo, and Sethuraman, 2020), and peer preferences (Cox, Fonseca, Pakzad-Hurson, and Pecenco, 2023; Leshno, 2022), among many others.

There is an active strand of empirical market design literature that utilizes randomization from school assignment lotteries for causal inference of school effectiveness. When restricting attention to applicants with the same preferences and priorities, the school assignment is purely decided by random lotteries, and hence, are uncorrelated with the applicants' potential outcomes at the assigned schools. The last condition is called unconfoundedness in the literature, and is sufficient for identifying the outcomes of interest. However, conditioning on preferences and priorities may result in low statistical power, since typically only a few applicants share the exact same preferences and priority groups. In their influential work, Abdulkadiroğlu et al. (2017) develop an estimation strategy based on conditioning on the school assignment probabilities, or the *propensity scores*, instead of preferences and priorities. As shown by Rosenbaum and Rubin (1983), the propensity score conditioning gives the coarsest partition that preserves the unconfoundedness condition. Hence, the method of Abdulkadiroğlu et al. (2017) fully exploits the quasi-experimental variation in the school assignment lotteries. The method has been used for estimating educational outcomes in a variety of contexts, by papers such as Angrist, Gray-Lobe, Idoux, and Pathak (2022); Angrist, Hull, Pathak, and Walters (2020); Dur, Hammond, Lenard, Morrill, Morrill, and Paeplow (2021); Hahm and Park (2021); Winters and Shanks (2021), among many others. The practical challenge with this method is that the school assignment probabilities are unknown. When rich preferences condition holds, I show that the observed proportions of different applicant types at different schools under the DA outcome are consistent estimators for the propensity scores, even with a single set of simulated random lotteries.

Another strand of empirical market design literature has been using the continuum model for demand analysis / preference estimation (Agarwal and Somaini, 2018; Artemov et al., 2023; Che, Hahm, and He, 2023). These methodologies too rely on a DA convergence and/or unique stable assignment results.

My results brings all the aforementioned theoretical and empirical analyses and methodologies to a more solid grounding.

3 Preliminaries

3.1 The Continuum Model

There is a unit mass of applicants $\mathcal{A}$ and a non-empty finite set of schools S. Each school $s \in S$ has a capacity $q_s \in \mathbb{R}_+$. Each applicant $a \in \mathcal{A}$ has a preference ranking (linear order) $\succ_a$ over $S \cup \{\emptyset\}$. Here, $\emptyset$ denotes the 'outside option' that represents a guaranteed school or the option of being unassigned. Let $\mathcal{P}$ denote the space of preference rankings. Each applicant a has school-specific priority scores $r_a \in [0,1]^S$. The economy is described by a (Borel) probability measure η over $\mathcal{P} \times [0,1]^S$. For a set of applicants $A \subseteq \mathcal{A}$, I abuse notation to write

$$\eta\big(A\big) = \eta\Big((\succ_a, r_a) : a \in A\Big),$$

whenever the latter set is measurable.

I assume strict priorities, that is, for any $s \in S$ and $x \in [0,1]$,

$$\eta\big(a \in \mathcal{A} : r_{as} = x\big) = 0.$$

The condition implies that η is non-atomic.

For a vector $c \in [0,1]^S$, the *demand set* $D_s(c) \subseteq \mathcal{A}$ at school s is defined as

$$D_s(c) = \Big\{a \in \mathcal{A} : r_{as} > c_s, s \succ_a \emptyset \text{ and } s \succ_a s' \text{ for all } s' \in S \setminus \{s\} \text{ with } r_{as'} > c_{s'}\Big\}.$$

In other words, the demand $D_s(c)$ at school s is the set of applicants whose priority scores at s exceed c_s, and who prefer school s to $\emptyset$, and who prefer s to every other school $s' \in S \setminus \{s\}$ where their priority scores exceed $c_{s'}$. In this case, we say that applicant a demands school s at vector c. Note that an applicant can demand at most one school.

An assignment $\mu = (\mu_s)_{s \in S}$ specifies a subset $\mu_s \subseteq \mathcal{A}$ for each school $s \in S$, such that

(i) $\mu_s \cap \mu_{s'} = \emptyset$ for all $s, s' \in S$ with $s \neq s'$,

(ii) μ_s is measurable and $\eta(\mu_s) \leq q_s$ for all $s \in S$.

Let $\mathcal{M}$ denote the set of all assignments.

For each $c \in [0,1]^S$ and $x \in [0,1]$, let $c(s,x) \in [0,1]^S$ denote the vector that potentially differs from c by that its s-coordinate is equal to x. That is, $c_s(s,x) = x$ and $c_{s'}(s,x) = c_{s'}$ for all $s' \in S \setminus \{s\}$. Consider the mapping $\mathcal{T} : [0,1]^S \to [0,1]^S$ such that for each $c \in [0,1]^S$ and $s \in S$,

$$\mathcal{T}_s(c) = \begin{cases} \inf \Big\{ x \in [0,1] : \eta\big(D_s(c(s,x))\big) \leq q_s \Big\} & \text{if } \eta\big(D_s(c)\big) \geq q_s, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a vector $c \in [0,1]^S$ is a *stable cutoff profile*, if it is a fixed point of $\mathcal{T}$. We say that $\mu \in \mathcal{M}$ is a *stable assignment* if there is a stable cutoff profile c such that $\mu_s = D_s(c)$ for all $s \in S$.

The applicant-proposing deferred acceptance (DA) is a procedure that outputs a stable assignment. Namely, DA starts with the vector $0 \in [0,1]^S$ and iteratively applies the mapping $\mathcal{T}$ as it converges to a stable cutoff profile. Formally, consider the sequence $(c(t))_{t \in \mathbb{N}_0}$ in $[0,1]^S$, where $c(0) = 0 \in [0,1]^S$ and $c(t) = \mathcal{T}(c(t-1))$ for all $t \geq 1$. Since $c(0) = 0$, by definition of $\mathcal{T}$, it is immediate that $c(1) \geq c(0)$. Since $\mathcal{T}$ is weakly increasing (Proposition A1 of AL), by an induction argument we can establish that $c(t) \geq c(t-1)$ for all $t \geq 1$. Thus, $(c(t))_{t \in \mathbb{N}_0}$ is a weakly increasing sequence. Since $(c(t))_{t \in \mathbb{N}_0}$ is also bounded, it is therefore convergent. Let $c := \lim_{t \to \infty} c(t)$. The vector c is a fixed point of $\mathcal{T}$ (Proposition A2 of AL), hence c is stable cutoff profile. The DA outcome is the corresponding stable assignment $\mu^{DA} = (\mu_s^{DA})_{s \in S} = (D_s(c))_{s \in S}$.

3.2 Random Finite Economies

Let η be a continuum economy. For each $n \in \mathbb{N}$, consider a finite set of applicants A^n whose preferences and priority scores are i.i.d. sampled according to the distribution η. Let the measure η^n over $\mathcal{P} \times [0,1]^S$ correspond to the events' relative frequencies. That is, for any $\succ \in \mathcal{P}$ and $R \subseteq [0,1]^S$,

$$\eta^n\Big((\succ, r) : r \in R\Big) = \frac{1}{n} \cdot \big|\big\{a \in A^n : \succ_a = \succ, r_a \in R\big\}\big|.$$

The probability measure η^n is called a size-n finite economy. Since η has strict priorities, in almost all realizations of η^n, no two applicants will have the same priority scores at any school. Without loss of generality, we will restrict attention to such realizations only.

In the finite model, stable assignments and the DA procedure are defined as in the previous section, with η^n instead of η. More specifically, consider the mapping $\mathcal{T}^n : [0,1]^S \to [0,1]^S$ such that for each $c \in [0,1]^S$ and $s \in S$,

$$\mathcal{T}^n_s(c) = \begin{cases} \inf\Big\{x \in [0,1] : \eta^n\big(D_s(c(s,x))\big) \leq q_s\Big\} & \text{if } \eta^n\big(D_s(c)\big) \geq q_s, \\ 0 & \text{otherwise.} \end{cases}$$

A stable cutoff profile c^n of the finite economy is a fixed point of $\mathcal{T}^n$. The corresponding stable assignment μ^n is defined by $\mu^n_s = D_s(c^n)$ for all $s \in S$.

Consider the sequence $(c^n(t))_{t \in \mathbb{N}_0}$ in $[0,1]^S$, where $c^n(0) = 0 \in [0,1]^S$ and $c^n(t) = \mathcal{T}^n(c^n(t-1))$ for all $t \geq 1$, and let $c^n := \lim_{t\to\infty} c^n(t)$. Note that the convergence happens after finitely many iterations, that is, $c^n(t+1) = c^n(t)$ for some $t \in \mathbb{N}_0$. Therefore, c^n is a fixed point of $\mathcal{T}^n$. The DA outcome is the corresponding assignment $\mu^{DA,n} = \big(\mu^{DA,n}_s\big)_{s \in S} = (D_s(c^n))_{s \in S}$.

The definitions of the DA and stable assignments for the finite economies coincide with the ones in the standard model of Gale and Shapley (1962).

4 An Example of the Non-Convergence of the DA

The DA outcome in a continuum economy may substantially differ from the DA outcome of any large enough finite economy, with a probability that is bounded away from zero. I demonstrate this with an example.[3]

There is a unit mass of applicants $\mathcal{A}$ and two schools s_1 and s_2, with $q_{s_1} = q_{s_2} = \frac{1}{3}$. First, I will give an intuitive and informal description of the continuum economy η, and then I will define it formally.

Suppose that half of the applicants reside in the neighborhood of s_1, and the other half reside in the neighborhood of s_2. All applicants that reside in the neighborhood of s_1 have higher priority scores at s_1 than all applicants that reside in the neighborhood of s_2. Similarly, all applicants that reside in the neighborhood of s_2 have higher priority scores at s_2 than all applicants that reside in the neighborhood of s_1.

Suppose that one third of the applicants prefer s_1 over s_2 over $\emptyset$, one third prefer s_2 over s_1 over $\emptyset$, and the remaining one third prefer $\emptyset$ over s_1 and s_2. The preferences are independent of the neighborhood status. The unspecified aspects of the economy can be arbitrary.

Formally, the continuum economy η is a probability measure on $\mathcal{P} \times [0,1]^2$, satisfying the following properties:

- $\eta\Big(a \in \mathcal{A} : r_{as_1} \geq \frac{1}{2} \text{ and } r_{as_2} < \frac{1}{2}\Big) = \frac{1}{2}$
- $\eta\Big(a \in \mathcal{A} : r_{as_1} < \frac{1}{2} \text{ and } r_{as_2} \geq \frac{1}{2}\Big) = \frac{1}{2}$.
- for any measurable $R \subseteq [0,1]^2$,

$$\eta\Big(a \in \mathcal{A} : s_1 \succ_a s_2 \succ_a \emptyset, r_a \in R\Big) = \eta\Big(a \in \mathcal{A} : s_2 \succ_a s_1 \succ_a \emptyset, r_a \in R\Big)$$

[3] Azevedo and Leshno (2016) use a similar example to show the multiplicity of stable assignments.

$$= \eta\Big(a \in \mathcal{A} : \emptyset \succ_a s_1, s_2, r_a \in R\Big) = \frac{1}{3} \cdot \eta\Big(a \in \mathcal{A} : r_a \in R\Big),$$

- otherwise, the economy η is arbitrary.

By the third bullet point above, when the cutoffs at both schools is zero, the mass of applicants that demand each school is $\frac{1}{3}$. Therefore, $c = 0$ is the stable cutoff profile corresponding to the DA outcome at economy η. At the DA outcome, all applicants are assigned to their most preferred schools.

I will now show that for all large enough finite economies, the DA outcome will substantially differ from that of the continuum one.

Consider a random finite economy η^n for some $n \in \mathbb{N}$. Let us partition the set of applicants A^n into three subsets X^n, Y^n, and Z^n, where

$$X^n := \Big\{a \in A^n : s_1 \succ_a s_2 \succ_a \emptyset\Big\},\ Y^n := \Big\{a \in A^n : s_2 \succ_a s_1 \succ_a \emptyset\Big\},$$

$$Z^n := \Big\{a \in A^n : \emptyset \succ_a s_1, s_2\Big\}.$$

Since A^n corresponds to n i.i.d. draws from η, the random variable $\eta^n(Z^n) \cdot n$ follows a Binomial distribution with a success probability $\eta\big(a \in \mathcal{A} : \emptyset \succ_a s_1, s_2\big) = \frac{1}{3}$. By the central limit theorem, we have that as n goes to infinity,

$$\Big(\eta^n(Z^n) - \frac{1}{3}\Big) \cdot \frac{3\sqrt{n}}{\sqrt{2}} \xrightarrow{d} N(0,1).$$

Consider the event,

$$\eta^n(Z^n) < \frac{1}{3}.$$

Since $\eta^n(Z^n)$ is asymptotically normally distributed with a mean $\frac{1}{3}$, the probability of the event $\eta^n\big(Z^n\big) < \frac{1}{3}$ converges to one half. That is, for any $\delta > 0$, there is an $N_\delta \in \mathbb{N}$, such that $n > N_\delta$ implies that the probability of $\eta^n\big(Z^n\big) < \frac{1}{3}$ is larger than $\frac{1}{2} - \delta$.

Suppose that $\eta^n(Z^n) < \frac{1}{3}$. By the previous paragraph, in large markets this happens with a probability close to one half. In what follows, I will show that the DA outcome for the economy η^n substantially differs from the DA outcome for the economy η.

Note that

$$\eta^n(Z^n) < \frac{1}{3} \text{ implies } \eta^n(X^n \cup Y^n) > \frac{2}{3}.$$

Let c^n be the cutoff profile of the DA outcome for the economy η^n. By definition of $\mathcal{T}^n$, we have that $\eta^n\big(D_s(c^n)\big) \leq q_s = \frac{1}{3}$ for $s \in \{s_1, s_2\}$. Thus,

$$\eta^n\big(D_{s_1}(c^n) \cup D_{s_2}(c^n)\big) = \eta^n\big(D_{s_1}(c^n)\big) + \eta^n\big(D_{s_2}(c^n)\big) \leq \frac{2}{3}.$$

Since

$$\eta^n(X^n \cup Y^n) > \frac{2}{3} \geq \eta^n\big(D_{s_1}(c^n) \cup D_{s_2}(c^n)\big),$$

there is an applicant in $a \in X^n \cup Y^n$ such that $a \notin D_{s_1}(c^n) \cup D_{s_2}(c^n)$. One of the following should hold: either (i) a resides in the neighborhood of s_1, that is, $r_{as_1} \geq \frac{1}{2}$ and $r_{as_2} < \frac{1}{2}$, or (ii) or a resides in the neighborhood of s_2. that is, $r_{as_1} < \frac{1}{2}$ and $r_{as_2} \geq \frac{1}{2}$. Without loss of generality, suppose that the former holds. Since $a \notin D_{s_1}(c^n) \cup D_{s_2}(c^n)$, and a prefers s_1 to $\emptyset$, it should be that $c^n_{s_1} \geq r_{as_1} \geq \frac{1}{2}$.

Let $\bar{A}^n$ be the set of applicants in X^n that reside in the neighborhood of s_2. That is,

$$\bar{A}^n := \Big\{a \in X^n : r_{as_1} < \frac{1}{2}\Big\}$$

Since $c^n_{s_1} \geq \frac{1}{2}$, none of the applicants in $\bar{A}^n$ is assigned to their most preferred school s_1.

Since one half of the applicants reside in the neighborhood of s_2, one third of the applicants prefer s_1 as their first choice, and the preferences and priority scores are independent, the probability that $\eta^n(\bar{A}^n)$ is close to $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ converges to one as n goes to infinity. This convergence in probability holds even after conditioning on the event that $\eta^n(Z^n) < \frac{1}{3}$. Thus, close to $\frac{1}{6}$ proportion of applicants are not assigned to their most preferred schools at the DA outcome of the finite economy. This outcome significantly differs from the DA outcome of the corresponding continuum economy.

To summarize, in the example above the DA outcome of any large enough finite economy substantially differs from the DA outcome of the corresponding continuum economy with a probability close to one half.

The intuition behind the non-convergence in the above example may be better grasped with the Gale-Shapley description of the DA. Consider the event where the set of applicants whose preferences are $s_1 \succ_a s_2 \succ_a \emptyset$ or $s_2 \succ_a s_1 \succ_a \emptyset$ is larger than the sum of capacities of s_1 and s_2. Then, in the first round of the Gale-Shapley procedure, some applicants with preferences $s_1 \succ_a s_2 \succ_a \emptyset$ or $s_2 \succ_a s_1 \succ_a \emptyset$ will apply to their most preferred schools, and some of them will be rejected by one of the schools. Without loss of generality, suppose that this school is s_1. Then, either all applicants who applied to s_1 have greater than $1/2$ priority scores at s_1, or some applicants with smaller than $1/2$ priority scores at s_1 will be rejected by the school. Suppose the latter holds. Then, the rejected applicants will apply to s_2. Again, either all applicants who applied to s_2 have greater than $1/2$ priority scores at s_2, or some applicants with smaller than $1/2$ priority scores at s_2 will be rejected by the school. Suppose the latter holds. The set of newly rejected applicants is disjoint from the previous one, since the previous set of rejected applicants have smaller than $1/2$ priority scores at s_1, and hence greater than $1/2$ priority scores at s_2. The newly rejected applicants will apply to schools s_1, and so on and so forth. This process will induce rejections in every round, until all applicants who apply to some school have greater than $1/2$ priority scores there. Thus, in a large enough market, this assignment will drastically differ from the continuum one where all applicants are assigned to their first choices (and hence, half of the applicants assigned to each school have smaller than $1/2$ priority scores there).

The convergence could be restored in this example if there was some (can be arbitrarily small) proportion of applicants with rankings $s_1 \succ_a \emptyset \succ_a s_2$ and $s_2 \succ_a \emptyset \succ_a s_1$ at each neighborhood. In that case, after every round of rejections, some proportion of applicants would opt out for the outside option. Hence, the Gale-Shapley procedure would conclude with almost everyone assigned to their most preferred choices. In the next section, I show that the convergence result always holds when the preferences are 'sufficiently rich'.

5 Main Results

Consider the family of sets $\mathcal{R} := \big\{(a_s, b_s)_{s\in S} : a, b \in [0,1]^S, a \le b\big\}$.

We will assume the relative ranking of the outside option are sufficiently represented in the economy in the following sense: for each $R \in \mathcal{R}$, if there is a positive mass of applicants whose priority scores are in R, and who prefer some school s to all schools in some subset of schools S', then there should be some mass of applicants whose priority scores are in R, and who prefer s to $\emptyset$, and $\emptyset$ to all schools in S'.

Definition 1. *The economy η has ϵ-**rich preferences** for some $\epsilon > 0$, if for any $R \in \mathcal{R}$, $s \in S$ and $S' \subseteq S \setminus \{s\}$,*

$$\eta\Big(a \in \mathcal{A} : r_a \in R,\ \textit{and } s \succ_a s' \textit{ for all } s' \in S'\Big) \ge \epsilon \implies$$

$$\eta\Big(a \in \mathcal{A} : r_a \in R,\ \textit{and } s \succ_a \emptyset \succ_a s' \textit{ for all } s' \in S'\Big) > 0$$

Definition 2. *The economy η has **rich preferences**, if for any $R \in \mathcal{R}$, $s \in S$ and $S' \subseteq S \setminus \{s\}$,*

$$\eta\Big(a \in \mathcal{A} : r_a \in R,\ \textit{and } s \succ_a s' \textit{ for all } s' \in S'\Big) > 0 \implies$$

$$\eta\Big(a \in \mathcal{A} : r_a \in R,\ \textit{and } s \succ_a \emptyset \succ_a s' \textit{ for all } s' \in S'\Big) > 0$$

The rich preferences assumption will be satisfied if the valuations for the outside option are sufficiently dispersed.[4] In school choice, the outside option may represent a private school, a charter school, homeschooling, a guaranteed school, or a school outside of the district. Applicants from any neighborhood could potentially rank some outside option in any position in their preference rankings. Most importantly, the condition does not impose restrictions on the relative ranking of schools (other than the outside option) or the priority scores.

Another simple condition that implies rich preferences is the following: for any $R \in \mathcal{R}$ that is represented in the economy, there are applicants that rank each school $s \in S$ as their first choice and $\emptyset$ as their second choice. That is, for any $R \subseteq \mathcal{R}$ with $\eta\big(a \in \mathcal{A} : r_a \in R\big) > 0$ and $s \in S$,

$$\eta\Big(a \in \mathcal{A} : r_a \in R, \text{ and } s \succ_a \emptyset \succ_a s' \text{ for all } s' \in S \setminus \{s\}\Big) > 0.$$

It is immediate that the above condition implies rich preferences.

The rich preferences condition is sufficient for the DA convergence. Before stating the main results, I will first introduce some more notation. Let us define a distance function on the space $S \times 2^S \times \mathcal{R}$ as follows. For any two elements $(s, S', R), (\tilde{s}, \tilde{S}', \tilde{R}) \in S \times 2^S \times \mathcal{R}$, where $R = (a_s, b_s)_{s \in S}$ and $\tilde{R} = (\tilde{a}_s, \tilde{b}_s)_{s \in S}$, the distance between (s, S', R)

[4]Formally, suppose an applicant a's preference ranking is determined by her cardinal valuations $v_a \in [0,1]^S$ for all schools S and her cardinal valuation $u_a \in [0,1]$ for the outside option $\emptyset$. Suppose the joint distribution of cardinal valuations and priority scores admits a density function f. Then, the corresponding continuum economy η will satisfy the rich preferences assumption if the conditional distributions of the outside option's valuation are fully supported. That is, for any $(x_1, x_2) \subseteq [0,1]$, and $y, z \in [0,1]^S$,

$$\int_0^1 f\big(u_a = x, v_a = y, r_a = z\big)\, dx > 0 \implies \int_{x_1}^{x_2} f\big(u_a = x, v_a = y, r_a = z\big)\, dx > 0.$$

and $(\tilde{s}, \tilde{S}', \tilde{R})$ is defined as

$$\mathbb{1}\big[(s, S') \neq (\tilde{s}, \tilde{S}')\big] + \sum_{s\in S} \big(|a_s - \tilde{a}_s| + |b_s - \tilde{b}_s|\big).$$

This distance metricizes the space $S \times 2^S \times \mathcal{R}$.

Suppose the economy has ϵ-rich preferences for some $\epsilon > 0$. Consider the set

$$\mathcal{C} := \Big\{(s, S', R) \in S \times 2^S \times \mathcal{R} : \eta\big(a \in \mathcal{A} : r_a \in R, \text{ and } s \succ_a s' \text{ for all } s' \in S'\big) \geq \epsilon\Big\}.$$

Since η is non-atomic, $\mathcal{C}$ is compact in the above metric space. By definition of $\mathcal{C}$ and ϵ-rich preferences, for every $(s, S', R) \in \mathcal{C}$, we have that

$$\eta\Big(a \in \mathcal{A} : r_a \in R, \text{ and } s \succ_a \emptyset \succ_a s' \text{ for all } s' \in S'\Big) > 0.$$

Since $\mathcal{C}$ is compact, there is a number $\bar{\lambda}_\epsilon > 0$, such that for every $(s, S', R) \in \mathcal{C}$, we have that

$$\eta\Big(a \in \mathcal{A} : r_a \in R, \text{ and } s \succ_a \emptyset \succ_a s' \text{ for all } s' \in S'\Big) \geq \bar{\lambda}_\epsilon.$$

Let $\lambda_\epsilon := \min\{\bar{\lambda}_\epsilon, \epsilon\}$.

Now I will formally state the DA convergence results. For two sets A and B, let their symmetric difference be $A \triangle B := (A \setminus B) \cup (B \setminus A)$. I define a distance function $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ between assignments in the economy η by

$$\mathcal{D}\big(\mu, \mu' \mid \eta\big) = \sum_{s\in S} \eta\Big(\mu_s \triangle \mu'_s\Big).$$

My first theorem gives an approximate convergence result for an economy that satisfies ϵ-rich preferences for some fixed $\epsilon > 0$.

Theorem 1. *Suppose an economy η has ϵ-rich preferences for some $\epsilon > 0$. Then, there is an $N_\epsilon \in \mathbb{N}$, such that for all $n > N_\epsilon$, the probability that*

$$\mathcal{D}\big(\mu^{DA}, \mu^{DA,n} \mid \eta^n\big) < |S| \cdot 2^{|S|+2} \cdot \epsilon,$$

is greater than

$$1 - 2^{|S+1|!} \cdot (2n^2 + n + 1)^{|S|} \cdot e^{-\frac{\lambda_\epsilon^2 \cdot 2 \cdot n}{|S|^2}}$$

The proof of Theorem 1 is in Appendix A.1. The convergence result and the rates of convergence are established using a well-known theorem by Vapnik and Chervonenkis (1971).

It is immediate from the definitions that an economy has rich preferences if and only if it has ϵ-rich preferences for every $\epsilon > 0$. Hence, the following theorem, that gives an exact convergence result, is a corollary of Theorem 1.

Theorem 2. *Suppose η has rich preferences. Then, for any $\epsilon > 0$, there is an $N_\epsilon \in \mathbb{N}$, such that for all $n > N_\epsilon$, the probability that*

$$\mathcal{D}\big(\mu^{DA}, \mu^{DA,n} \mid \eta^n\big) < |S| \cdot 2^{|S|+2} \cdot \epsilon,$$

is greater than

$$1 - 2^{|S+1|!} \cdot (2n^2 + n + 1)^{|S|} \cdot e^{-\frac{\lambda_\epsilon^2 \cdot 2 \cdot n}{|S|^2}}$$

Uniqueness of stable assignments or the 'small core' property (that is, a small diameter of a set of stable assignments) is a commonly observed phenomenon in real-life assignment problems. Many papers have provided theoretical justifications for this observation using different large matching models. In the continuum model, sufficiency conditions for unique stable assignments have been provided by AL. As I discussed in the introduction, those conditions might be restrictive in several applications.[5] I show that the rich preferences condition is sufficient for the uniqueness of stable assignments and the small core result.

Theorem 3. *If η has ϵ-rich preferences for some $\epsilon > 0$, then for any stable assignments μ and $\tilde{\mu}$ of η,*

$$\mathcal{D}\big(\mu, \tilde{\mu} \mid \eta\big) < |S| \cdot 2^{|S|} \cdot \epsilon.$$

[5]Sufficiency conditions in other large market models have been given by papers like Immorlica and Mahdian (2005) and Ashlagi, Kanoria, and Leshno (2017). In a recent paper, Arnosti (2022) studies a model that generalizes many existing large market models. The discussions of these alternative models are beyond the scope of this paper.

Moreover, if η has rich preferences, then for any stable assignments μ and $\tilde{\mu}$ of η,

$$\mathcal{D}\big(\mu, \tilde{\mu} \mid \eta\big) = 0.$$

The proof of Theorem 3 is in Appendix A.2.

6 Application: Propensity Score Estimation

Consider the continuum model, and suppose that applicants belong to a finite set of types. Each applicant $a \in \mathcal{A}$ has a type $(\succ_a, \rho_a) \in \mathcal{P} \times \{1, 2, \ldots, K\}^S$, where $\succ_a$ denotes the applicant's preference ranking over $S \cup \{\emptyset\}$ and $\rho_{as} \in \{1, 2, \ldots, K\}$ denotes her *priority group* at the school. For example, it is typical for school districts to partition applicants into priority groups based on sibling or walk-zone status. In that case, we would have $K = 4$, and

$$\rho_{as} = \begin{cases} 4 & \text{if } a \text{ resides in the neighborhood of } s \text{ and has a sibling at the school,} \\ 3 & \text{if } a \text{ does not reside in the neighborhood of } s \text{ and has a sibling at the school,} \\ 2 & \text{if } a \text{ resides in the neighborhood of } s \text{ and does not have a sibling at the school,} \\ 1 & \text{otherwise.} \end{cases}$$

Let $p(\succ, \rho) \in [0, 1]$ denote the proportion of applicants with a type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$. Each applicant $a \in A$ has a score

$$r_{as} = \rho_{as} + l_{as}$$

at school $s \in S$, where l_{as} is a i.i.d uniform random draw from the unit interval (i.e., a lottery number).

Formally, the economy above can be described by a distribution η, such that for all $\succ \in \mathcal{P}$, $\rho \in \{1, 2, \ldots, K\}^S$, and $[b_{1s}, b_{2s}]_{s \in S} \subseteq \big[\rho_s, \rho_s + 1\big]_{s \in S}$,

$$\eta\Big(a \in \mathcal{A} : \succ_a = \succ, r_a \in [b_{1s}, b_{2s}]_{s \in S}\Big) = p(\succ, \rho) \cdot \prod_{s \in S} \big(b_{2s} - b_{1s}\big).$$

Observation 1. *The economy* η *has* ϵ-***rich preferences*** *if and only if for any* $\rho \in \{1, 2, \ldots, K\}^S$, $s \in S$, *and* $S' \subseteq S \setminus \{s\}$,

$$\eta\big(a \in \mathcal{A} : \rho_a = \rho, \text{ and } s \succ_a s' \text{ for all } s' \in S'\big) \geq \epsilon \implies$$

$$\eta\big(a \in \mathcal{A} : \rho_a = \rho, \text{ and } s \succ_a \emptyset \succ_a s' \text{ for all } s' \in S'\big) > 0$$

Observation 2. *The economy* η *has* ***rich preferences*** *if and only if for any* $\rho \in \{1, 2, \ldots, K\}^S$, $s \in S$, *and* $S' \subseteq S \setminus \{s\}$,

$$\eta\big(a \in \mathcal{A} : \rho_a = \rho, \text{ and } s \succ_a s' \text{ for all } s' \in S'\big) > 0 \implies$$

$$\eta\big(a \in \mathcal{A} : \rho_a = \rho, \text{ and } s \succ_a \emptyset \succ_a s' \text{ for all } s' \in S'\big) > 0$$

For each $n \in \mathbb{N}$, consider a random finite economy η^n where the applicants' preference rankings and priority scores are i.i.d sampled according to η. Consider the probability $P_s^n(\succ, \rho)$ that an applicant with type $(\succ, \rho)$ is assigned to a school s. As discussed in the introduction, estimating these probabilities is crucial for propensity score analysis. The reader can learn the details of this methodology in Abdulkadiroğlu et al. (2017).

A potential practical challenge with the propensity score conditioning method of Abdulkadiroğlu et al. (2017) is that type-specific school assignment probabilities are unknown. I show that when the rich preferences condition holds, the observed proportions of different types of applicants to different schools under DA are consistent estimators for the corresponding assignment probabilities. Moreover, it is sufficient to compute these proportions for just a single simulated economy, and one does not need to repeatedly draw thousands or millions of random economies to consistently estimate the DA school assignment probabilities as in Dur et al. (2021) and Winters and Shanks (2021). The result provides a theoretical justification for estimating the propensity scores consistently and (computationally) efficiently, with a single simulation.

Consider a type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$ with $\eta\big(a \in \mathcal{A} : \succ_a = \succ, \rho_a = \rho\big) = p(\succ, \rho) > 0$. By the law of large numbers, the probability that $\eta^n\big(a \in A^n : \succ_a = \succ, \rho_a = \rho\big) > 0$ converges to one as n goes to infinity. Whenever $\eta^n\big(a \in A^n : \succ_a = \succ, \rho_a = \rho\big) > 0$, we can define

$$\hat{P}_s^n(\succ, \rho) := \frac{\eta^n\Big(a \in \mu_s^{DA,n} : \succ_a = \succ, \rho_a = \rho\Big)}{\eta^n\Big(a \in A^n : \succ_a = \succ, \rho_a = \rho\Big)},$$

as the proportion of type $(\succ, \rho)$ applicants at school s under the DA outcome $\mu^{DA,n}$ of η^n.

Proposition 1. *Suppose that η has ϵ-rich preferences for some $\epsilon > 0$. Consider a type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$ with $p(\succ, \rho) > 0$. Then, the probability that*

$$\big|\hat{P}_s^n(\succ, \rho) - P_s^n(\succ, \rho)\big| < \frac{1}{p(\succ, \rho)} \cdot \big(|S| \cdot 2^{|S|+3} + 1\big) \cdot \epsilon$$

converges to one, as n goes to infinity.

Proof. Let μ^{DA} be the DA outcome for the economy η. Consider an arbitrary type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$ with $p(\succ, \rho) > 0$, and let

$$P_s(\succ, \rho) := \frac{\eta\Big(a \in \mu_s^{DA} : \succ_a = \succ, \rho_a = \rho\Big)}{\eta\Big(a \in \mathcal{A} : \succ_a = \succ, \rho_a = \rho\Big)},$$

denote the proportion of type $(\succ, \rho)$ applicants at school s at μ^{DA}.

By the law of large numbers, $\hat{P}_s^n$ is well-defined with probability close to one, for any sufficiently large n. Moreover, by Theorem 2, when n is sufficiently large, the probability that

$$\big|\hat{P}_s^n(\succ, \rho) - P_s(\succ, \rho)\big| < \frac{1}{p(\succ, \rho)} \cdot |S| \cdot 2^{|S|+2} \cdot \epsilon,$$

is larger than $1 - \epsilon$.

Note that $P_s^n(\succ, \rho)$ is the expectation of $\hat{P}_s^n(\succ, \rho)$. Since the probability that $\hat{P}_s^n(\succ, \rho)$ is less than $\frac{1}{p(\succ,\rho)} \cdot |S| \cdot 2^{|S|+2} \cdot \epsilon$ apart from $P_s(\succ, \rho)$ is close to one, when n is sufficiently large, it should be that the expectation $P_s^n(\succ,\rho)$ of $\hat{P}_s^n(\succ,\rho)$ is also close to $P_s(\succ,\rho)$ with probability close to one. That is, for any sufficiently large n,

$$\left|P_s(\succ,\rho)-P_s^n(\succ,\rho)\right| < (1-\epsilon)\cdot\frac{1}{p(\succ,\rho)}\cdot\left(|S|\cdot 2^{|S|+2}\cdot\epsilon\right)+\epsilon\cdot 1 < \frac{1}{p(\succ,\rho)}\cdot\left(|S|\cdot 2^{|S|+2}+1\right)\cdot\epsilon.$$

Combining the results in the previous two paragraph, and by the triangle inequality, we get that for any sufficiently large n, the probability that

$$\left|\hat{P}_s^n(\succ,\rho)-P_s^n(\succ,\rho)\right| < \frac{1}{p(\succ,\rho)}\cdot\left(|S|\cdot 2^{|S|+3}+1\right)\cdot\epsilon,$$

is close to one. This completes the proof of Proposition 2. □

Proposition 2. *Suppose that η has rich preferences. Consider a type $(\succ,\rho)\in\mathcal{P}\times\{1,2,...,K\}^S$ with $p(\succ,\rho)>0$. Then, for any $\epsilon>0$, the probability that*

$$\left|\hat{P}_s^n(\succ,\rho)-P_s^n(\succ,\rho)\right| < \frac{1}{p(\succ,\rho)}\cdot\left(|S|\cdot 2^{|S|+3}+1\right)\cdot\epsilon$$

converges to one, as n goes to infinity.

Proof. The result directly follows from Proposition 1. □

Proposition 2 says that $\hat{P}_s^n$ is a consistent estimator for type-specific school assignment probabilities. Since there are finitely many types, the convergence results hold uniformly across all types.

When the rich preferences condition fails, one may still provide informative bounds on type-specific school assignment probabilities using Proposition 1. This may facilitate meaningful empirical analysis like in Rosenbaum (2005).

References

Abdulkadiroğlu, A., J. D. Angrist, Y. Narita, and P. A. Pathak (2017): "Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation," *Econometrica*, 85, 1373–1432.

Abdulkadiroğlu, A. and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2015): "Expanding "Choice" in School Choice," *American Economic Journal: Microeconomics*, 7, 1–42.

Agarwal, N. and P. Somaini (2018): "Demand Analysis Using Strategic Reports: An Application to a School Choice Mechanism," *Econometrica*, 86(2), 391–444.

Angrist, J., G. Gray-Lobe, C. M. Idoux, and P. A. Pathak (2022): "Still Worth the Trip? School Busing Effects in Boston and New York," Workin Paper No. w30308. National Bureau of Economic Research.

Angrist, J., P. Hull, P. A. Pathak, and C. R. Walters (2020): "Simple and Credible Value-Added Estimation Using Centralized School Assignment," National Bureau of Economic Research, Working paper No. w28241.

Arnosti, N. (2022): "A continuum model of stable matching with finite capacities," arXiv preprint arXiv:2205.12881.

Artemov, G., Y.-K. Che, and Y. He (2023): "Stable Matching with Mistaken Agents," *Journal of Political Economy Microeconomics*, 1, 270–320.

Ashlagi, I., Y. Kanoria, and J. D. Leshno (2017): "Unbalanced Random Matching Markets: The Stark Effect of Competition," *Journal of Political Economy Microeconomics*, 125, 69–98.

Azevedo, E. M. and J. D. Leshno (2016): "A Supply and Demand Framework for Two-sided Matching Markets," *Journal of Political Economy*, 124, 1235–1268.

Balinski, M. and T. Sönmez (1999): "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, 84, 73–94.

Bodoh-Creed, A. L. (2020): "Optimizing for Distributional Goals in School Choice

Problems," *Management Science*.

Celebi, O. and J. P. Flynn (2021): "Priority Design in Centralized Matching Markets," Working Paper.

Che, Y.-K., D. W. Hahm, and Y. He (2023): "Leveraging Uncertainties to Infer Preferences: Robust Analysis of School Choice," Working Paper.

Cox, N., R. Fonseca, B. Pakzad-Hurson, and M. Pecenco (2023): "Peer Preferences in Centralized School Choice Markets: Theory and Evidence," Working Paper.

Dur, U., R. G. Hammond, M. A. Lenard, M. Morrill, T. Morrill, and C. Paeplow (2021): "The Attraction of Magnet Schools: Evidence from Embedded Lotteries in School Assignment," Working paper.

Escobar, J. F. and L. Huerta (2021): "School Choice and Segregation," Working paper.

Feigenbaum, I., Y. Kanoria, I. Lo, and J. Sethuraman (2020): "Efficient Seat Reassignment after Late Cancellations," *Management Science*, 66, 5341–5361.

Gale, D. and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Hahm, D. W. and M. Park (2021): "A Dynamic Framework of School Choice: Effects of Middle Schools on High School Choice," Working paper.

Immorlica, N. and M. Mahdian (2005): "Marriage, Honesty, and Stability," *SODA*, 53–62.

Leshno, J. (2022): "Stable Matching with Peer-Dependent Preferences in Large Markets: Existence and Cutoff Characterization," Workin Paper No. w30308. National Bureau of Economic Research.

Leshno, J. D. and I. Lo (2021): "The Cutoff Structure of Top Trading Cycles in School Choice," *The Review of Economic Studies*, 88, 1582–1623.

Rao, R. R. (1962): "Relations Between Weak and Uniform Convergence of Measures with Applications," *The Annals of Mathematical Statistics*, 659–680.

Rosenbaum, P. R. (2005): "Sensitivity Analysis in Observational Studies," *Encyclopedia of Statistics in Behavioral Science*.

Rosenbaum, P. R. and D. B. Rubin (1983): "The Central Role of the Propensity Score in Observational Studies for Causal Effects," *Biometrica*, 70, 41–55.

Roth, A. E. (1984): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 92, 991–1016.

——— (1986): "On the Allocation of Residents to Rural Hospitals: A General Property of Two-sided Matching Markets," *Econometrica*, 425–427.

Vapnik, V. and A. Chervonenkis (1971): "On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities," *Theory of Probability and Its Applications*, 16, 264–280.

Winters, M. A. and C. Shanks (2021): "The Effect of Attending a Charter School in Newark, New Jersey, on Student Test Scores," Educational Evaluation and Policy Analysis.

A Proofs of Main Results

A.1 Proof of Theorem 1

Consider the family of sets

$$\mathcal{F} = 2^{\mathcal{P}} \times \mathcal{R}.$$

The following lemma establishes that the differences in the measures of all sets in $\mathcal{F}$ under η and η^n uniformly converge to zero in probability. This lemma is an application of a well-known result on uniform laws of large numbers for empirical processes.

Lemma 1. *Fix an $\epsilon > 0$. Then, for any $n \geq \frac{8 \cdot |S|^2}{\lambda_\epsilon^2}$, the probability that*

$$\sup_{F \in \mathcal{F}} \left\{ \left| \eta^n\Big((\succ, r) \in F\Big) - \eta\Big((\succ, r) \in F\Big) \right| \right\} < \frac{1}{2 \cdot |S|} \cdot \lambda_\epsilon,$$

is greater than

$$1 - 2^{|S+1|!} \cdot (2n^2 + n + 1)^{|S|} \cdot e^{-\frac{\lambda_\epsilon^2 \cdot 2 \cdot n}{|S|^2}}.$$

Proof. I first introduce several definitions.

Consider a size-n sample from $\mathcal{P} \times [0,1]^S$, denoted by $\mathcal{X} = \{\chi_1, \chi_2, \ldots, \chi_n\}$. We say a set $F \in \mathcal{F}$ induces a subset $\mathcal{X}'$ of $\mathcal{X}$, if $F \cap \mathcal{X} = \mathcal{X}'$. The index of the family $\mathcal{F}$ at the sample $\mathcal{X}$, denoted by $\Delta^{\mathcal{F}}(\mathcal{X})$, is the maximum number of subsets of X that can be induced by the sets in $\mathcal{F}$. The growth function of the family $\mathcal{F}$ evaluated at n, denoted by $\Pi^{\mathcal{F}}(n)$, is the largest possible index of $\mathcal{F}$ across all possible size-n samples from $\mathcal{P} \times [0,1]^S$. That is,

$$\Pi^{\mathcal{F}}(n) = \max\Big\{\Delta^{\mathcal{F}}(\mathcal{X}) : \mathcal{X} \subseteq \mathcal{P} \times [0,1]^S, |\mathcal{X}| = n\Big\}.$$

Using the definition of the growth function, we can establish that

$$\Pi^{\mathcal{F}}(n) = 2^{|S+1|!} \cdot \Big(\frac{n(n+1)}{2} + 1\Big)^{|S|}.$$

The result now follows from Theorem 2 of Vapnik and Chervonenkis (1971). More specifically, the constant $2^{|S+1|!} \cdot (2n^2 + n + 1)^{|S|}$ in front of the exponential function in Lemma 1 is the growth function evaluated at $2n$. $\square$

In what follows I will assume that η^n satisfies

$$\sup_{F \in \mathcal{F}} \left\{ \left| \eta^n \Big((\succ, r) \in F \Big) - \eta \Big((\succ, r) \in F \Big) \right| \right\} < \frac{1}{2 \cdot |S|} \cdot \lambda_\epsilon.$$

By Lemma 1, for any sufficiently large n, this happens with a probability close to one.

To complete the proof of Theorem 1, we need to establish that

$$\mathcal{D}\big(\mu^{DA}, \mu^{DA,n} \mid \eta^n\big) < |S| \cdot 2^{|S|+2} \cdot \epsilon.$$

Suppose, for the sake of contradiction, that

$$\mathcal{D}\big(\mu^{DA}, \mu^{DA,n} \mid \eta^n\big) \geq |S| \cdot 2^{|S|+2} \cdot \epsilon.$$

Let c and c^n be the cutoff profiles of assignments μ^{DA} and $\mu^{DA,n}$, respectively. Let A be the set of applicants who prefer their demanded school at c to their demanded school at c^n. Similarly, let A^n be the set of applicants who prefer their demanded school at c^n to their demanded school at c. Note that $2 \cdot \eta^n\big(A \cup A^n\big) \geq \mathcal{D}\big(\mu^{DA}, \mu^{DA,n} \mid \eta^n\big) \geq |S| \cdot 2^{|S|+2} \cdot \epsilon$, which implies that $\eta^n\big(A \cup A^n\big) \geq |S| \cdot 2^{|S|+1} \cdot \epsilon$. Hence, one of the following is true: (1) $\eta^n(A) \geq |S| \cdot 2^{|S|} \cdot \epsilon$, or (2) $\eta^n(A^n) \geq |S| \cdot 2^{|S|} \cdot \epsilon$. We study these two cases separately.

Case 1. Suppose $\eta^n(A) \geq |S| \cdot 2^{|S|} \cdot \epsilon$.

Let $\bar{S} := \big\{ s \in S : c_s < c_s^n \big\}$. The set is non-empty since $\eta^n(A) \geq |S| \cdot 2^{|S|} \cdot \epsilon > 0$.

For each $s \in S$, let $A_s \subseteq A$ be the subset of applicants in A who demand school s at cutoff profile c. Note that $A_s = \emptyset$ for all $s \notin \bar{S}$, that is, no applicant $a \in A$ demands a school $s \notin \bar{S}$ at c. To see this, for the sake of contradiction suppose that an applicant $a \in A$ demands a school $s \notin \bar{S}$ at c. Then, by definition of $\bar{S}$, we have that $c_s^n \leq c_s$. Therefore, at c^n, the applicant $a \in A$ demands a school that is weakly more preferred than s, contradicting the definition of set A.

Since $A_s = \emptyset$ for all $s \notin \bar{S}$, then $A = \cup_{s \in \bar{S}} A_s$.

Let $\bar{s} \in \arg\max_{s \in \bar{S}} \eta^n(A_s)$. Since $A = \cup_{s \in \bar{S}} A_s$, and $\eta^n(A) \geq |S| \cdot 2^{|S|} \cdot \epsilon$, we have that $\eta^n(A_{\bar{s}}) \geq 2^{|S|} \cdot \epsilon$.

By definition of A, and by $A_{\bar{s}} \subseteq A$, it should be that all applicants in $A_{\bar{s}}$ prefer their demanded school at c (i.e. $\bar{s}$) to their demanded school at c^n. Moreover, it should be that all applicants in $A_{\bar{s}}$ have their priority scores for $\bar{s}$ in the interval $(c_{\bar{s}}, c_{\bar{s}}^n]$, that is, $r_{a\bar{s}} \in (c_{\bar{s}}, c_{\bar{s}}^n]$ for all $a \in A_{\bar{s}}$.

For a given subset of schools $S' \subseteq S \setminus \{\bar{s}\}$, let

$$A_{\bar{s},S'} := \left\{a \in \mathcal{A} : r_{a\bar{s}} \in (c_{\bar{s}}, c_{\bar{s}}^n],\ \bar{s} \succ s \text{ for all } s \in S',\ \text{and } r_{as} > c_s^n \text{ only if } s \in S'\right\}.$$

Note that $A_{\bar{s}} \subseteq \cup_{S' \subseteq S \setminus \{\bar{s}\}} A_{\bar{s},S'}$. Let $\tilde{S} \in \arg\max_{S' \subseteq S \setminus \{\bar{s}\}} \eta^n(A_{\bar{s},S'})$. Then, $\eta^n(A_{\bar{s}}) \geq 2^{|S|} \cdot \epsilon$ implies that $\eta^n(A_{\bar{s},\tilde{S}}) \geq 2 \cdot \epsilon$.

Since $\sup_{F \in \mathcal{F}} \left\{\left|\eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big)\right|\right\} < \frac{\lambda_\epsilon}{2 \cdot |S|} < \epsilon$, and $\eta^n(A_{\bar{s},\tilde{S}}) \geq 2 \cdot \epsilon$, we have that $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$.

Let

$$A(\bar{s}) := \Big\{a \in \mathcal{A} : r_{a\bar{s}} \in (c_{\bar{s}}, c_{\bar{s}}^n],\ \bar{s} \succ \emptyset \succ s \text{ for all } s \in \tilde{S},\ \text{and } r_{as} > c_s^n \text{ only if } s \in \tilde{S}\}\Big\}.$$

Since η has ϵ-rich preferences and $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$, we have that $\eta(A(\bar{s})) \geq \lambda_\epsilon$. Again, since $\sup_{F \in \mathcal{F}} \left\{\left|\eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big)\right|\right\} < \frac{\lambda_\epsilon}{2 \cdot |S|}$, and $\eta(A(\bar{s})) \geq \lambda_\epsilon$, we conclude that $\eta^n(A(\bar{s})) > \eta(A(\bar{s})) - \frac{\lambda_\epsilon}{2 \cdot |S|} \geq \lambda_\epsilon - \frac{\lambda_\epsilon}{2 \cdot |S|} \geq \frac{\lambda_\epsilon}{2}$.

By definition of $\bar{S}$, we have that $c_s^n > c_s \geq 0$ for all $s \in \bar{S}$. Thus, $\eta^n(D_s(c^n)) = q_s$ for all $s \in \bar{S}$. Let $\bar{A} := \cup_{s \in \bar{S}} D_s(c^n)$ be the set of applicants who demand a school in $\bar{S}$ at cutoff profile c^n. Since $c_s < c_s^n$ for all $s \in \bar{S}$, and $c_s \geq c_s^n$ for all $s \notin \bar{S}$, it should be that all applicants in $\bar{A}$ demand some school in $\bar{S}$ also at c. Therefore,

$$\bar{A} \subseteq \cup_{s \in \bar{S}} D_s(c).$$

By definition of $A(\bar{s})$, we have that $A(\bar{s}) \subseteq D_{\bar{s}}(c)$. Moreover, again by definition of $A(\bar{s})$, all these applicants prefer $\emptyset$ to every school s where their priority score exceeds the cutoff c_s^n. Hence, $A(\bar{s}) \cap \big(\cup_{s \in S} D_s(c^n)\big) = \emptyset$. Since $A(\bar{s}) \cap \big(\cup_{s \in S} D_s(c^n)\big) = \emptyset$ and $\bar{A} = \cup_{s \in \bar{S}} D_s(c^n) \subseteq \cup_{s \in S} D_s(c^n)$, we have that $A(\bar{s}) \cap \bar{A} = \emptyset$.

Therefore,

$$\sum_{s \in \bar{S}} \eta^n\big(D_s(c)\big) \geq \eta^n(\bar{A}) + \eta^n(A(\bar{s})) > \sum_{s \in \bar{S}} q_s + \frac{\lambda_\epsilon}{2}.$$

Let $\hat{s} \in \arg\max_{s \in \bar{S}} \eta^n\big(D_s(c)\big) - q_s$. Since $\sum_{s \in \bar{S}} \eta^n\big(D_s(c)\big) > \sum_{s \in \bar{S}} q_s + \frac{\lambda_\epsilon}{2}$, it should be that $\eta^n\big(D_{\hat{s}}(c)\big) > q_{\hat{s}} + \frac{\lambda_\epsilon}{2 \cdot |S|}$.

Since $\sup_{F \in \mathcal{F}} \big\{\big|\eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big)\big|\big\} < \frac{\lambda_\epsilon}{2 \cdot |S|}$, we get that

$$\eta\big(D_{\hat{s}}(c)\big) > \eta^n\big(D_{\hat{s}}(c)\big) - \frac{\lambda_\epsilon}{2 \cdot |S|} > q_{\hat{s}} + \frac{\lambda_\epsilon}{2 \cdot |S|} - \frac{\lambda_\epsilon}{2 \cdot |S|} = q_{\hat{s}}.$$

This contradicts that c is a stable cutoff profile of the economy η.

Case 2. The study of Case 2 is similar to Case 1, but it is not completely symmetric.[6]

Suppose $\eta^n(A^n) \geq |S| \cdot 2^{|S|} \cdot \epsilon$.

Let $\bar{S} := \big\{s \in S : c_s^n < c_s\big\}$. The set is non-empty since $\eta^n(A^n) \geq |S| \cdot 2^{|S|} \cdot \epsilon > 0$.

For each $s \in S$, let $A_s \subseteq A^n$ be the subset of applicants in A^n who demand school s at cutoff profile c^n. Note that $A_s = \emptyset$ for all $s \notin \bar{S}$. To see this, for the sake

[6] In both cases, we start with assumption on $\eta^n(A)$ or $\eta^n(A^n)$, but not on $\eta(A)$ or $\eta(A^n)$. Hence, there is an asymmetry in the proofs of the two cases.

of contradiction suppose that an applicant $a \in A^n$ demands a school $s \notin \bar{S}$ at c^n. Then, by definition of $\bar{S}$, we have that $c_s \leq c_s^n$. Therefore, at c, the applicant $a \in A^n$ demands a school that is weakly more preferred than s, contradicting the definition of set A^n. Since $A_s = \emptyset$ for all $s \notin \bar{S}$, we have that $A^n = \cup_{s \in \bar{S}} A_s$.

Let $\bar{s} \in \arg\max_{s \in \bar{S}} \eta^n(A_s)$. Since $A^n = \cup_{s \in \bar{S}} A_s$, and $\eta^n(A^n) \geq |S| \cdot 2^{|S|} \cdot \epsilon$, it should be that $\eta^n(A_{\bar{s}}) \geq 2^{|S|} \cdot \epsilon$.

By definition of A^n, and by $A_{\bar{s}} \subseteq A^n$, we have that all applicants in $A_{\bar{s}}$ prefer their demanded school at c^n (that is, $\bar{s}$) to their demanded school at c. Moreover, it should be that all applicants in $A_{\bar{s}}$ have their priority scores for $\bar{s}$ in the interval $(c_{\bar{s}}^n, c_{\bar{s}}]$, that is, $r_{a\bar{s}} \in (c_{\bar{s}}^n, c_{\bar{s}}]$ for all $a \in A_{\bar{s}}$.

For a given subset of schools $S' \subseteq S \setminus \{\bar{s}\}$, let

$$A_{\bar{s},S'} := \Big\{a \in \mathcal{A} : r_{a\bar{s}} \in (c_{\bar{s}}^n, c_{\bar{s}}],\ \bar{s} \succ s \text{ for all } s \in S',\ \text{and } r_{as} > c_s \text{ only if } s \in S'\Big\}.$$

Note that $A_{\bar{s}} \subseteq \cup_{S' \subseteq S \setminus \{\bar{s}\}} A_{\bar{s},S'}$. Let $\tilde{S} \in \arg\max_{S' \subseteq S \setminus \{\bar{s}\}} \eta^n(A_{\bar{s},S'})$. Then, $\eta^n(A_{\bar{s}}) \geq 2^{|S|} \cdot \epsilon$ implies that $\eta^n(A_{\bar{s},\tilde{S}}) \geq 2 \cdot \epsilon$.

Since $\sup_{F \in \mathcal{F}} \big\{\big|\eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big)\big|\big\} < \frac{\lambda_\epsilon}{2 \cdot |S|} < \epsilon$, and $\eta^n(A_{\bar{s},\tilde{S}}) \geq 2 \cdot \epsilon$, we have that $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$.

Let

$$A(\bar{s}) := \Big\{a \in \mathcal{A} : r_{a\bar{s}} \in (c_{\bar{s}}^n, c_{\bar{s}}],\ \bar{s} \succ \emptyset \succ s \text{ for all } s \in \tilde{S},\ \text{and } r_{as} > c_s \text{ only if } s \in \tilde{S}\}\Big\}.$$

Since η has ϵ-rich preferences and $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$, we have that $\eta(A(\bar{s})) \geq \lambda_\epsilon$.

By definition of $\bar{S}$, we have that $c_s > c_s^n \geq 0$ for all $s \in \bar{S}$. Thus, $\eta(D_s(c)) = q_s$ for all $s \in \bar{S}$. Let $\bar{A} := \cup_{s \in \bar{S}} D_s(c)$ be the set of applicants who demand a school in $\bar{S}$ at cutoff profile c. Since $c_s^n < c_s$ for all $s \in \bar{S}$, and $c_s^n \geq c_s$ for all $s \notin \bar{S}$, it should be that all applicants in $\bar{A}$ demand some school in $\bar{S}$ also at c^n. Therefore,

$$\bar{A} \subseteq \cup_{s \in \bar{S}} D_s(c^n).$$

By definition of of $A(\bar{s})$, we have that $A(\bar{s}) \subseteq D_{\bar{s}}(c^n)$. Moreover, again by definition of $A(\bar{s})$, all these applicants prefer $\emptyset$ to every school s where their priority score exceeds the cutoff c_s. Hence, $A(\bar{s}) \cap \big(\cup_{s \in S} D_s(c)\big) = \emptyset$. Since $A(\bar{s}) \cap \big(\cup_{s \in S} D_s(c)\big) = \emptyset$ and $\bar{A} = \cup_{s \in \bar{S}} D_s(c) \subseteq \cup_{s \in S} D_s(c)$, we have that $A(\bar{s}) \cap \bar{A} = \emptyset$.

Therefore,

$$\sum_{s \in \bar{S}} \eta\big(D_s(c^n)\big) \geq \eta(\bar{A}) + \eta(A(\bar{s})) \geq \sum_{s \in \bar{S}} q_s + \lambda_\epsilon > \sum_{s \in \bar{S}} q_s + \lambda_\epsilon$$

Let $\hat{s} \in \arg\max_{s \in \bar{S}} \eta\big(D_s(c^n)\big) - q_s$. Since $\sum_{s \in \bar{S}} \eta\big(D_s(c^n)\big) \geq \sum_{s \in \bar{S}} q_s + \lambda_\epsilon$, it should be that $\eta\big(D_{\hat{s}}(c^n)\big) \geq q_{\hat{s}} + \frac{\lambda_\epsilon}{|S|}$.

Since $\sup_{F \in \mathcal{F}} \big\{ \big| \eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big) \big| \big\} < \frac{\lambda_\epsilon}{2 \cdot |S|}$, we get that

$$\eta^n\big(D_{\hat{s}}(c^n)\big) > \eta\big(D_{\hat{s}}(c^n)\big) - \frac{\lambda_\epsilon}{2 \cdot |S|} \geq q_{\hat{s}} + \frac{\lambda_\epsilon}{|S|} - \frac{\lambda_\epsilon}{2 \cdot |S|} > q_{\hat{s}}.$$

This contradicts that c^n is a stable cutoff profile of the economy η^n, completing the proof of Theorem 1 is complete.

Remark: In the proof above, we only used the fact that c and c^n are cutoff profiles of η and η^n, respectively. Thus, Theorem 1 result can be generalized to establish convergence of arbitrary stable assignments μ^n of η^n to an arbitrary stable assignment μ of η. Namely, for a sufficiently large n, with probability close to one,

$$\mathcal{D}\big(\mu, \mu^n \mid \eta^n\big) < |S| \cdot 2^{|S|+2} \cdot \epsilon,$$

where μ and μ^n are arbitrary stable assignments of economies η and η^n, respectively.

A.2 Proof of Theorem 3

The remark in the last paragraph of the previous section can be used to establish establish the 'small core' result. Instead, I give a different/direct proof that establishes the convergence under a tighter bounds stated in Theorem 3.

The second part of the Theorem 3 follows from the first one. In what follows I will prove the first part of the theorem.

Suppose η has ϵ-rich preferences for some $\epsilon > 0$. Let μ and $\tilde{\mu}$ be arbitrary stable assignments of η, and let c and $\tilde{c}$ be the corresponding cutoff profiles. Without loss of generality, by taking $\tilde{c} := c \wedge \tilde{c}$ and $c := c \vee \tilde{c}$ if necessary, we can assume that $\tilde{c} \leq c$.[7]

Suppose, for the sake of contradiction, that

$$\mathcal{D}\big(\mu, \tilde{\mu} \mid \eta\big) = \sum_{s \in S} \eta\big(\mu_s \triangle \tilde{\mu}_s\big) = \sum_{s \in S} \eta\big(D_s(c) \triangle D_s(\tilde{c})\big) \geq |S| \cdot 2^{|S|} \cdot \epsilon.$$

Consider the set $A := \cup_{s \in S} D_s(c) \triangle D_s(\tilde{c})$, that is, the set of applicants who demand different schools at cutoff profiles c and $\tilde{c}$.

Note that

$$\sum_{s \in S} \eta\big(D_s(c) \triangle D_s(\tilde{c})\big) = 2 \cdot \eta\big(A\big).$$

Hence, $\sum_{s \in S} \eta\big(D_s(c) \triangle D_s(\tilde{c})\big) \geq |S| \cdot 2^{|S|} \cdot \epsilon$ implies that $\eta\big(A\big) \geq |S| \cdot 2^{|S|-1} \cdot \epsilon$.

Let

$$A_s := \big\{a \in \mathcal{A} : \ a \in D_s(\tilde{c}) \text{ and } a \notin D_s(c)\big\}.$$

That is, A_s is the set of applicants who demand s at cutoff profile $\tilde{c}$ and do not demand s at cutoff profile c. Note that, $A = \cup_{s \in S} A_s$. Let $\bar{s} \in \arg\max_{s \in S} \eta(A_s)$. Then, $\eta(A) \geq |S| \cdot 2^{|S|-1} \cdot \epsilon$ implies that $\eta(A_{\bar{s}}) \geq 2^{|S|-1} \cdot \epsilon$.

Since $\tilde{c} \leq c$, all applicants in $A_{\bar{s}}$ prefer their demanded school at cutoff profile $\tilde{c}$ (that is, $\bar{s}$) to their demanded school at cutoff profile c. Moreover, it should be that all applicants in $A_{\bar{s}}$ have their priority scores for $\bar{s}$ in the interval $(\tilde{c}_{\bar{s}}, c_{\bar{s}}]$, that is, $r_{a\bar{s}} \in (\tilde{c}_{\bar{s}}, c_{\bar{s}}]$ for all $a \in A_{\bar{s}}$.

[7] Here, $c \wedge \tilde{c}$ and $c \vee \tilde{c}$ denote the coordinate-wise minimum and coordinate-wise maximum of c and $\tilde{c}$, respectively. By the lattice structure of the set of stable assignments (Azevedo and Leshno, 2016), c and $\tilde{c}$ will both be stable cutoff profiles. Moreover, the distance between the redefined stable assignments will be weakly larger than the distance between the original ones.

For a given subset of schools $S' \subseteq S \setminus \{s\}$, let

$$A_{\bar{s},S'} := \big\{a \in \mathcal{A} : r_{a\bar{s}} \in (\tilde{c}_{\bar{s}}, c_{\bar{s}}],\ \bar{s} \succ s \text{ for all } s \in S',\ \text{and } r_{as} > c_s \text{ only if } s \in S'\big\}.$$

Note that $A_{\bar{s}} \subseteq \cup_{S' \subseteq S\setminus\{\bar{s}\}} A_{\bar{s},S'}$. Let

$$\tilde{S} \in \underset{S' \subseteq S\setminus\{\bar{s}\}}{\arg\max}\, \eta(A_{\bar{s},S'}).$$

Then, $\eta(A_{\bar{s}}) \geq 2^{|S|-1} \cdot \epsilon$ implies that $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$.

Let

$$A(\bar{s}) = \big\{a \in \mathcal{A} : r_{a\bar{s}} \in (\tilde{c}_{\bar{s}}, c_{\bar{s}}],\ \bar{s} \succ \emptyset \succ s' \text{ for all } s \in \tilde{S},\ \text{and } r_{as} > c_s \text{ only if } s \in \tilde{S}\big\}$$

Since η has ϵ-rich preferences, and $\eta(A_{\bar{s},\tilde{S}}) \geq \epsilon$, we conclude that $\eta(A(\bar{s})) > 0$.

Since all applicants in $A(\bar{s})$ prefer $\bar{s}$ to $\emptyset$, and their priority scores exceed $\tilde{c}_{\bar{s}}$, it should be that none of these applicants are assigned to the outside option $\emptyset$ at cutoffs $\tilde{c}$. That is, $A(\bar{s}) \subseteq \cup_{s\in S} D_s(\tilde{c})$. Moreover, all applicants in $A(\bar{s})$ prefer $\emptyset$ to every school s where their priority score exceeds the cutoff c_s. Hence, all these applicants are assigned to the outside option at cutoffs c. That is, $A(\bar{s}) \cap \big(\cup_{s\in S} D_s(c)\big) = \emptyset$. This is a contradiction to the rural hospital theorem (Azevedo and Leshno, 2016; Roth, 1986), which says that the set of applicants that are assigned to the outside option is the same across all stable assignments. This completes the proof of Theorem 3.

B Additional Results

B.1 Continuity of Stable Assignments

In this section, I establish the following general continuity result for stable assignments.

Theorem 4. *Suppose η is absolutely continuous and has ϵ-rich preferences for some $\epsilon > 0$. Let $(\eta^n)_{n\in\mathbb{N}}$ be a sequence of economies that converges to η in a weak sense. Then, there is an $N_\epsilon \in \mathbb{N}$, such that for all $n > N_\epsilon$,*

$$D\big(\mu, \mu^n \mid \eta^n\big) < |S| \cdot 2^{|S|+2} \cdot \epsilon,$$

where μ and μ^n are arbitrary stable assignments of η and η^n, respectively.

Proof. The weak convergence of $(\eta^n)_{n\in\mathbb{N}}$ to η, implies that

$$\sup_{F\in\mathcal{F}} \left\{ \left| \eta\Big((\succ, r) \in F\Big) - \eta^n\Big((\succ, r) \in F\Big) \right| \right\} \longrightarrow 0.$$

This follows from Theorem 4.2 in Rao (1962). Hence, the convergence above implies that there is an $N_\epsilon \in \mathbb{N}$, such that

$$\sup_{F\in\mathcal{F}} \left\{ \left| \eta^n\big((\succ, r) \in F\big) - \eta\big((\succ, r) \in F\big) \right| \right\} < \frac{1}{2 \cdot |S|} \cdot \lambda_\epsilon$$

is satisfied for all $n > N_\epsilon$. The remainder of the proof of Theorem 4 is identical to that of Theorem 1. □

B.2 Alternative Estimators for School Assignment Probabilities

Following Abdulkadiroğlu et al. (2017) and Agarwal and Somaini (2018), I provide alternative consistent estimators for type-specific school assignment probabilities. The estimators can be computed analytically, using the observed cutoff profile of the DA outcome of a single simulated random finite economy.

Proposition 3. *Suppose η has rich preferences. For any $n \in \mathbb{N}$, let c^n be the DA cutoff profile for η^n, and for any type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$, let*

$$\tilde{P}^n_s(\succ, \rho) := \prod_{s':s'\succ s} \Big(\mathbb{1}\big[\rho_{s'} = \lfloor c^n_{s'} \rfloor\big] \cdot \big(c^n_{s'} - \lfloor c^n_{s'} \rfloor\big) + \mathbb{1}\big[\rho_{s'} < \lfloor c^n_{s'} \rfloor\big]\Big)$$

$$\cdot \Big(\mathbb{1}\big[\rho_s = \lfloor c_s^n \rfloor\big] \cdot \big(\lfloor c_s^n \rfloor + 1 - c_s^n\big) + \mathbb{1}\big[\rho_s > \lfloor c_s^n \rfloor\big]\Big).$$

Then, $\tilde{P}_s^n(\succ, \rho)$ converges to $P_s^n(\succ, \rho)$ (with a probability arbitrarily close to one).

Proof. In the proof of Proposition 2, it has been established that $\hat{P}_s^n(\succ, \rho)$ converges both to $P_s(\succ, \rho)$ and to $P_s^n(\succ, \rho)$. Hence, it should be that $P_s^n(\succ, \rho)$ converges to $P_s(\succ, \rho)$.

I will show that $\tilde{P}_s^n(\succ, \rho)$ converges to $P_s(\succ, \rho)$, which would imply that $\tilde{P}_s^n(\succ, \rho)$ to $P_s^n(\succ, \rho)$.

Let c denote the DA cutoff profile of the continuum economy η. Then, by the definition of the DA, and the description of η, for all $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$,

$$P_s(\succ, \rho) = \prod_{s' : s' \succ s} \Big(\mathbb{1}\big[\rho_{s'} = \lfloor c_{s'} \rfloor\big] \cdot \big(c_{s'} - \lfloor c_{s'} \rfloor\big) + \mathbb{1}\big[\rho_{s'} < \lfloor c_{s'} \rfloor\big]\Big)$$

$$\cdot \Big(\mathbb{1}\big[\rho_s = \lfloor c_s \rfloor\big] \cdot \big(\lfloor c_s \rfloor + 1 - c_s\big) + \mathbb{1}\big[\rho_s > \lfloor c_s \rfloor\big]\Big).$$

By Theorem 2, c^n converges to c. Therefore, $\tilde{P}_s^n(\succ, \rho)$ converges to $P_s(\succ, \rho)$. This completes the proof of Proposition 3. □

As mentioned by Agarwal and Somaini (2018), dimensionality reduction is the main benefit of this alternative estimation method, compared to the one in the main text. More specifically, to obtain $\tilde{P}_s^n(\succ, \rho)$ for all types $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$, one needs to estimate the cutoff profile of the DA outcome and apply the formula in Proposition 3 for all applicants and schools in their preference lists. In contrast, to obtain the school assignment probabilities using the method in the main text, one might need to compute $\hat{P}_s^n(\succ, \rho)$ all type $(\succ, \rho) \in \mathcal{P} \times \{1, 2, ..., K\}^S$ represented in the finite economy, and for each school s that is ranked by at least one applicant of this type. Both methods are computationally tractable.