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Chaos and unpredictability with time inconsistent policy makers

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We analyze the existence of equilibria with complex dynamics in a policy framework with time inconsistency. We consider an economy where, in each period, the policy maker in power determines the level of a durable public good (or bad) that creates strategic linkages across policy periods. When the decision-making process is time consistent—such as when a benevolent planner sets policy in every period—the economy exhibits a unique equilibrium where the state converges to a deterministic steady state. When the identity of the decision maker changes probabilistically over time as in a political equilibrium, the decision-making process becomes time inconsistent. In this scenario, we identify conditions under which equilibria with cycles of more than two periods and chaotic dynamics can emerge. Depending on the economy's fundamental parameters, these equilibria may produce ergodic distributions in which the state variable either persistently overshoots the planner's steady state or fluctuates around it. The extent of chaotic behavior is influenced by the degree of time inconsistency: as the degree of time inconsistency approaches zero, the size of the support of the ergodic distribution converges to zero as well.

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JEL classification. C61, C62, C73, D78.

1. Introduction

Environments characterized by dynamic inconsistencies are pervasive in economics. They not only include settings with time-inconsistent preferences, but also many natural strategic interactions in which the players have "standard" exponential preferences, such as voluntary contribution games and common pool problems, various political economy problems such as positive models of public debt, or models of durable public goods.[1] In dynamic political economy games, time inconsistency in decision making emerges from the fact that decision makers change over time and have heterogeneous

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[1] For examples of models of common pool problems or contribution games, see Levhari and Mirman (1980), Fershtman and Nitzan (1991), Marx and Matthews (2000), Battaglini, Nunnari, and Palfrey (2014),

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preferences; in voluntary contribution and common pool games, time inconsistency stems from externalities in the individual decisions of multiple decision makers.[2]

Despite the importance of these problems, there is still a limited understanding of the strategic behavior in stochastic games with time inconsistency. In applied work, the focus has been either on simplified environments such as, for example, models with finite horizons or models in which the players have dominant strategies, where equilibria are more readily characterized; or on environments in which well-behaved equilibria exist and can be relied upon to make predictions. The equilibria in these games often look like solutions to a constrained planner's problems, in which the state variable monotonically converges to a steady state—albeit too low as in dynamic public good games, or too high, as in political economy models of public debt.[3]

In this paper, we study a simple dynamic policy game with time inconsistency and we ask whether it can generate equilibria with complex dynamics in generic economies, since most existing models that generate complex dynamics rely on nongeneric conditions. Specifically, we aim at characterizing conditions under which time inconsistency allows for the existence of equilibria with cycles (with possibly arbitrarily long periods) and equilibria in which the state variables follow chaotic trajectories. In the latter equilibria, the state variable can be considered unpredictable because it is highly sensitive to the initial state: two identical economies, starting from arbitrarily close but different initial conditions, can diverge significantly in the long term in ways that are difficult to predict, even in the absence of shocks. If the initial state is observed with noise, as is naturally assumed, then long-term behavior remains unpredictable, even if the noise is arbitrarily small and there are no stochastic shocks to the economies.

To analyze the issue, we study an infinite-horizon game in which an incumbent policy maker selects the level of a durable public good (or bad) that strategically links policy making periods. We study the political equilibrium of this economy assuming that two parties alternate in power as in the classic game by Alesina and Tabellini (1990).[4]

In the absence of time inconsistency, the economy has a unique equilibrium, and this equilibrium has simple dynamics in which the state variable monotonically converges to a unique steady state. When policies are selected in a political equilibrium, however, the set of equilibria is very different and equilibria with complex dynamics may emerge. We first focus the analysis to the case in which preferences are quasilinear.

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among others. For political economy models of public debt, see Persson and Svensson (1989), Alesina and Tabellini (1990), Battaglini and Coate (2008), and Yared (2010). For models of durable public goods, see Battaglini and Coate (2007).

[2]In all the example presented above, a decision maker at $t+1$ faces a different intertemporal rate of substitution between $t+1$ and $t+2$ than a decision maker at t. A more detailed discussion of the relationship between these models and single agent models with time inconsistent preferences will be presented in Section 5.2.

[3]Models in which political equilibria can be characterized as solutions of constrained planner's problems have been provided by, among others, Battaglini and Coate (2008), Klein, Krusell, and Rios-Rull (2008), Yared (2010).

[4]As we explain in detail below, the logic of the model can be applied with minor changes to other economic environments, such as single agent consumption savings decision problems with quasi-hyperbolic discounting; dynamic free rider problems, such as voluntary contribution and common pool games; or more sophisticated political economy games with noncooperative bargaining.

We demonstrate that, regardless of the degree of time inconsistency, Markov equilibria with persistent cycles and chaotic dynamics exist if the relative importance of the public good component is sufficiently high or, ceteris paribus, if the degree of time inconsistency is not excessively large. We then extend the analysis to a more general class of utility functions.

In the equilibria with chaos, starting from almost any initial condition, the state variable converges to a region with positive measure and then "wanders around" in its inside, following a deterministic but aperiodic trajectory. Under general conditions, the state variable behaves like a random variable with a continuous distribution, in the sense that the ergodic distribution of the state variable converges to an absolutely continuous distribution on the set (a phenomenon often referred to as "ergodic chaos"). For some equilibria, this absolutely continuous distribution can be characterized in closed form as a function of the parameters of the economy (such as the degree of time inconsistency and the importance of the public good).

Existence and the properties of these equilibria are intimately connected to the presence of time inconsistency. The size of the set in which the state wanders around depends on the degree of time inconsistency: as time inconsistency converges to zero, chaotic equilibria continue to exist, but the size of the set in which the state wanders around shrinks to zero. A similar phenomenon occurs for equilibria with cycles: as time inconsistency converges to zero, the distance between the states in the periods converges to zero. This implies that depending on the degree of time inconsistency, equilibria with complex dynamics can be consistent with a variety of phenomena: from describing environments with large, possibly cyclical policy swings, when time inconsistency is large; to environments with small, noisy perturbations around a steady state generating a "fog of predictions" around an expected long term value, when time inconsistency is small. Failure to converge to a deterministic outcome in these equilibria highlights a novel source of inefficiency, distinct from the typical inefficiencies highlighted in the literature, where steady states are usually characterized as having too little of a good thing or too much of a bad thing. We show that when there are equilibria with persistent cycles, or equilibria with chaos, we do not necessarily have a simple "one-dimensional" bias. It is possible to construct equilibria under which the state of the economy fluctuates around the planner's optimum. This outcome is even worse than reaching a constant steady state equal to the inefficient expected value since preferences are concave in the state variable.

A limitation of the results described above is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but it is instead a feature of the specific class of equilibria whose existence we prove. Our results can be collectively interpreted as an *impossibility result*: for the simple yet natural economy we consider, it is impossible to predict equilibrium behavior in the sense that there are always chaotic equilibria that make deterministic predictions impossible even in the absence of shocks.

The paper contributes to three main lines of research: the economic literature on complex dynamics and chaos; the literature on dynamic decision with time inconsistent decision makers; and the literature in political economy and public economics in which time inconsistency emerges in equilibrium despite players having standard exponential preferences.

The question of whether dynamic economic models can generate complex dynamics and chaos has been studied extensively in the 1980s and 1990s. Identifying natural examples of economic environments with robust chaotic dynamics occurring in equilibrium, however, has proven to be an elusive task. Early economic examples of models with complex dynamics relied on a sufficient condition proposed by Li and Yorke (1975) that, while relatively easy to establish, did not exclude the typical case in which trajectories with complex dynamics are unstable and reachable only from a measure zero of initial conditions, making cycles with more than 2 periods or chaotic behavior unobservable (this form of unobservable chaotic behavior is sometimes refereed to as topological chaos).[5] Economic examples with more robust forms of complex dynamics, such as the ergodic chaos mentioned above, have been presented in the subsequent literature. These applications, however, relied on sufficient conditions that require the difference equation describing the dynamics to assume specific functional forms: piecewise linear maps, typically "V" or "inverse V" shaped; or piecewise smooth, expansive maps (i.e., functions with nondifferentiable "spikes" and derivative larger than one in absolute value on both sides of the "spike"). These properties are not naturally derived for optimal investment functions, except if specific technologies, or exogenous constraints are imposed, such as credit constraints or other constraints that force the state variable to become nonmonotonic.[6] While there are environments in which these assumptions are appropriate, our results rely neither on the assumptions nor on the techniques used in these works. Alternatively, the literature allows for smooth difference equations for which it is easier to provide a microeconomic foundations (S-unimodal maps), but only for nongeneric parametrizations. Our contribution relies on the construction of a novel class of equilibria, one that is large enough to select an equilibrium with the right properties for complex dynamics, in each parametrization of the environment. The characterization, in turn, is a consequence of the presence of time inconsistency.

The second literature to which our paper contributes is the literature on dynamic decision with time inconsistency. Phelps and Pollak (1968) started this literature by characterizing the linear Markov equilibrium in a single agent model of intertemporal consumption allocation. A general characterization of the Euler equations in similar

[5] Early seminal work include Day (1982), Benhabib and Day (1982), Grandmont (1985), who study overlapping generation models; Boldrin and Montrucchio (1986), Matsuyama (1999); Bewley (1986) and Woodford (1988) who study models with market imperfections. A critique of the notion of topological chaos used in some of the early literature is presented by Grandmont (1985) and Melese and Transue (1986). More recent related work includes Khan and Mitra (2005), GRSS+ (2022), and Mignot, Tramontana, and Westerhoff (2023). See Majumdar, Mitra, and Nishimura (2000), Hommes (2013), Rosser (2021), and Bischi, Cerboni Baiardi, Lamantia, and Radi (2024) for surveys of this literature and its implications for economics.

[6] Economic examples with these properties have been presented, for example, by Day and Shafer (1987), Day and Pianigiani (1991), Deneckere and Judd (1992), and more recently, Matsuyama, Sushko, and Gardini (2016). In these examples, the state x_t evolves according to a system $x_{t+1} = y(x_t)$ in which $y(\cdot)$ is assumed either piecewise linear and "V" shaped; or to be "expansive," i.e., satisfying $\inf(y'(x)) > 1$, so that if there is a maximum, it occurs in a non-differentiable "spike." These results rely on sufficient conditions first presented by Lasota and Yorke (1973) and/or its subsequent refinements.

problems is presented in Harris and Laibson (2001), and analyses of saving dynamics for a representative agent with quasi-geometric preferences are presented by Morris and Postlewaite (1997), Chatterjee and Eyigungor (2016), Cao and Werning (2018a,b), among others. Krusell and Smith (2003) also studied the issue of multiplicity of equilibria in these problems, and highlighted indeterminacy due to multiple possible equilibrium steady states. The problem we study in this paper is different: while we also obtain multiple equilibria, our main result is the proof of the existence of equilibria with complex, deterministic dynamics. In these equilibria, unpredictability occurs for a given equilibrium, not because of multiplicity.

Finally, our work contributes to the study of applied models where time inconsistency emerges in equilibrium despite players having standard exponential preferences. We have already cited above works on voluntary contribution and common pool games and political economy. As mentioned, this literature has mostly focused on well-behaved equilibria or environments in which the state converges to a deterministic steady state. Among the exceptions, Boylan, Ledyard, and McKelvey (1996) and Bai and Lagunoff (2011) study the problem in a political economy setting, while Battaglini, Nunnari, and Palfrey (2012) study voluntary contributions to a public good. Boylan, Ledyard, and McKelvey (1996) consider a model in which simple cycles with finite orbit may emerge when the policy maker selects policies that can be defeated by the smallest possible majority, and s/he can commit for at least 3 periods into the future; the length of the commitment period determines the length of the cycle in this model.[7] Bai and Lagunoff (2011) study a dynamic political game in which policies at t affect political turnover at $t+1$. They show conditions under which the equilibrium may converge to a stable steady state following a dampened cycle. Battaglini, Nunnari, and Palfrey (2012) study a model of free riding in which n agents independently contribute to a public good: using numerical examples, they show the existence of Markov equilibria with dampened cycles and with cycles of period 2.[8] None of these papers, nor to our knowledge any other in the political economy literature, has examined the emergence of complex cycles with period longer than 2 and/or chaotic behavior.[9]

2. Model

Consider an economy in which two parties alternate in power, call them A and B. Each party is associated to a constituency of citizens. We assume that there is a continuum of

[7]The authors also show that in their model no cycles are possible if the policy makers can not commit to a policy.

[8]The NBER working paper Battaglini, Nunnari, and Palfrey (2012) was published as Battaglini, Nunnari, and Palfrey (2014), but some of the results on cyclical equilibria were omitted in the 2014 version.

[9]A different (and less related to our work) body of research is the literature on the so called "political business cycles." This literature looks at models in which fluctuations in economic activity are generated by recurrent stimuli right before an election by an incumbent attempting to signal his/her competence to influence the electoral outcome; or right after an election as the uncertainty on the type of the winning party is resolved. These are typically stationary models with no underlying state variable, in which fluctuations coincide with the electoral cycle, not with a long term evolution of a state variable. See Alesina (1988), among others, for a survey.

citizens and we normalize the size of the constituency of each party to one. The party in power at time t selects a policy p_t from a set of feasible policies. The policy generates immediate costs or benefits for the citizens, and it also contributes to a long term state variable x_t that also affects the citizens' utility. For example, p_t may be a polluting activity that generates economic benefits to all (or a subset of) the citizens, yet it increases the stock of pollution (as measured by x_t).

The state variable x_t takes values in the real line $\mathbb{R}$, and it evolves according to: $x_{t+1} = (1-\gamma)x_t + p_t$, where γ is the rate of depreciation of x_t.[10] The policy p_t takes values in the set $P = [-l, \infty)$ for some $l > 0$, and thus it must be such that the state x_{t+1} satisfies $x_{t+1} \geq (1-\gamma)x_t - l$. The lower bound reflects the fact that there may be limits on the feasibility of a reduction in the state x.

In every period t, party $j \in \{A, B\}$ has a probability 1/2 to be in power. This assumption reflects the idea that the constituencies of the two parties have the same size, so the identity of the majority party at t is determined by chance. Citizens are assumed to be identical, except for the party whose constituency they belong to. Define $u^{i,j}(p, x)$ as the indirect utility function of a citizen in the constituency of party i when the party in office is j, the state at the beginning of the period is x, and the policy is $p = y - (1-\gamma)x$, where y is the state at the end of the period. A policy p can be interpreted as the expenditure on local public goods, subsidies, or other policies that the party in office can target to its constituency. Utility $u^{i,j}(p, x)$ depends on the party in power j because even if the levels of expenditure is the same for the two parties, the policy mix chosen by each party would naturally be different. The function $u^{i,j}(p, x)$ is assumed to be weakly concave in p, strictly concave in x, and continuously differentiable in both terms, with derivative with respect to the rth component equal to $u_r^{i,j}(\cdot)$ for $r = 1, 2$. We assume $u_1^{i,i}(\cdot) > u_1^{i,j}(\cdot)$ and $u_1^{i,i}(\cdot) > 0$: citizens' marginal utility for policies targeted to them is positive, and they derive higher utility from policies targeted to them than from policies targeted to the other constituency. The spillover of policies by policy maker j (targeted to district j) on district i can be positive (i.e., $u_1^{i,j}(\cdot) > 0$), as in the case of a highway or a bridge; or negative (i.e., $u_1^{i,j}(\cdot) < 0$), as in the case of a polluting power plant that benefits only to j's constituency, but that yet pollutes the air of both i's and j's constituencies. A specific example of these indirect utility functions is described below.

In this economy, an allocation is described by an infinite sequence x^∞ where $x^\infty = (x_0, \dots, x_t, \dots)$ and x_0 is exogenously given. The intertemporal utility at $t = 1$ of an agent in party i's constituency is $U^i(x^\infty) = \sum_{t=1}^{\infty} \delta^{t-1}[u^{i,\iota(t)}(x_t - (1-\gamma)x_{t-1}, x_{t-1})]$, where δ is the discount factor, $\iota(t)$ is the incumbent party in office at time t.

In Section 3, we study the case in which preferences are quasilinear and separable in x and p. Specifically, we assume the per period utility function:

$$u^{i,j}(p, x) = \alpha_{i,j} K \cdot p - e(x), \tag{1}$$

[10] Depending on the interpretation of the model x_t may be interpreted as a stock of capital that may generate positive or negative externalities, or just the stock of pollution (see Section 3.3 for a more extensive discussion of this point). In the second case, γ should be interpreted as the regeneration rate (i.e., the natural decline in the stock of pollutant).

where K is a strictly positive constant and $e(x)$ is strictly convex in x.[11] We assume that $\alpha_{i,i} = \alpha_{j,j} = 1$ and $\alpha_{i,j} = \alpha < 1$ when $i \neq j$: so, for a member of constituency i, the marginal utility of p is 1 when i's party is in office, and it is strictly lower than 1 otherwise. The parameter α is a direct measure of the time inconsistency generated by the political alternation of power. When $\alpha = 1$, there is no political conflict and no time inconsistency, since the policies of the two parties have the same effects on all citizens.[12] When $\alpha \in [0, 1)$, the policy benefits both the constituency of the party in power and the constituency of the party out of power, albeit less for the latter if $\alpha < 1$. When $\alpha \in (-\infty, 0)$, instead, the policy benefits the constituency of the party in power, but generates negative externalities for the rest of the citizens.[13] Later in this section and then in Section 3, we also assume a quadratic cost function:

$$e(x) = (\beta/2)(x - \widehat{x})^2, \tag{2}$$

where we assume $\beta > 0$ and $\widehat{x} \geq 0$. We will relax this assumption in Section 4.

We focus the analysis on symmetric Markov perfect equilibria, in which the parties use the same strategy, and in each period t these strategies are time-independent functions of the state x_t. Non-Markovian strategies will be discussed in Section 4.[14] A Markovian strategy is a function $p(\cdot)$, where $p(x)$ is the policy of the party in power when the state is x. Once $p(\cdot)$ is defined, then the state variable at $t+1$ is automatically defined as: $y : x \to (1 - \gamma)x + p(x)$. In the following, it will be more convenient to define equilibria in terms of $y(\cdot)$. We refer to $y(\cdot)$ as the *investment function*. Associated with any investment function y is a value function v, which specifies the expected discounted future payoff to an agent when the state is x.

An investment function y and an initial state x_0 define a dynamical system in which $x_{t+1} = y(x_t)$. We are interested in studying the dynamics that can emerge in equilibrium. It is worth stressing that the dynamics of the state in a symmetric equilibrium is deterministic and fully determined by the equilibrium y. The two parties in power alternate in power with probability 1/2, but they adopt the same strategy y in equilibrium, so the evolution of x does not depend on the outcome of the election but only on the initial condition x_0 and the shape of the investment function $y(\cdot)$.[15] Define $[y]^1(\cdot) = y(\cdot)$

[11]When $e(x)$ is increasing in x, then $-e(x)$ may be interpreted as a cost generated by x (e.g., pollution cost generated by the state x). But $e(x)$ is not necessarily increasing as, for example, when (2) with $\widehat{x} > 0$ is assumed. This implies that the model allows a variety of interpretations as it will be discussed in Section 3.3.

[12]The fact that with $\alpha = 1$ we do not have time inconsistency does not imply that policies are Pareto efficient: even with $\alpha = 1$, the policy maker in office ignores the externality on the constituency of the other policy maker.

[13]The indirect utility function (1) has a simple microfoundation. Assume that there are two possible policies: p^A that generates a marginal utility K on party A's constituency and αK on B's constituency; and a symmetric p^B that generates a marginal utility K on party B's constituency and αK on A's constituency. In a Markov equilibrium, for any level of expenditure p, party i would spend all in p^i, implying (1).

[14]The main result of our analysis is in proving the existence of equilibria with cycles, and/or unpredictable and chaotic behavior. The focus on Markov equilibria therefore is without loss of generality and makes the results stronger as it relies on simpler strategies.

[15]The fact that parties alternate in power stochastically is important only to the extent that it generates dynamic time inconsistency. In Section 5.2, I present an alternative decision model with a single decision maker with hyperbolic discounting and no shocks, as an example in which time inconsistency emerges even in the absence of shocks. More examples are presented in the working paper (Battaglini (2023)).

and, for any $k \geq 1$, $[y]^k(\cdot) = y([y]^{k-1}(\cdot))$. For any starting condition x_0, $[y]^k$ naturally defines a trajectory $\{x_1, \dots x_k, x_{k+1}, \dots\}$ in which $x_k = [y]^k(x_0)$. A cycle of period τ is a set $\{x_1, \dots, x_\tau\}$ such that $x_k = [y]^\tau(x_k)$ for all $k = 1, .., \tau$; any element of a cycle with τ element is called a periodic point of period τ. The simplest, and most widely studied, case of cycle is a *steady state*, which is a cycle of period 1. We define a cycle $\{x_1, \dots, x_\tau\}$ to be attracting (or asymptotically stable) if for all points x_k in the cycle there exists an open neighborhood U of x_k such that for all $x \in U$, we have $[y]^{m\tau}(x) \in U$ for any integer $m \geq 1$ and $\lim_{m\to\infty}[y]^{m\tau}(x) = x_k$. When a cycle is attracting, a small perturbation to the state variable does not alter the long run behavior of the system.

The mathematical literature has offered various definition of "chaotic" behavior of deterministic dynamical systems. The intuitive features of a "chaotic" system in a set I are that: (a) the system is invariant in I; (b) it has an aperiodic trajectory dense in I; and it has sensitive dependence, meaning that even an arbitrarily change in the initial condition leads to a large deviation in the long term. Dynamical systems with these properties are said to be incomputable or unpredictable in the sense that they give different predictions for arbitrarily close initial conditions (see, e.g., Devaney (1989)).

A standard formal definition of chaotic behavior is provided by Devaney (1989). We say that an investment function y is *transitive* in I if for any open $U, V \subset I$, there exists a k such that $[y]^k(U) \cap V \neq \emptyset$. Intuitively, a topologically transitive map "wanders" in the invariant set I, moving under iteration from one arbitrarily small neighborhood to any other. An investment function y exhibits *topological chaos* in a set I if it is transitive and it has a set of periodic points that is dense in I. If the two conditions of this definition are satisfied, then it can be shown that y is sensitive on initial conditions in the sense that there exists a χ such that, for any $x \in I$ and any neighborhood N of x there exists a $z \in N$ and a $m \geq 0$ such that $|[y]^m(x) - [y]^m(z)| > \chi$.[16]

A dynamical system is said to display *ergodic chaos* if the system is ergodic and the unique invariant distribution of the Perron–Frobenius operator is absolutely continuous with respect to the Lebesgue measure. This definition implies that, starting from a generic initial condition, the orbit described by y "fills up" the entire support of the ergodic distribution, and thus defines extremely complex dynamics. As we will see, our equilibria will satisfy both the topological and the ergodic definition of chaos. We will formally define and discuss ergodic chaos in Section 3.2.

Before studying equilibrium behavior in the model described above, it is useful to characterize the optimal policy when it is selected by a utilitarian planner (henceforth, the planner) under Assumptions (1) and (2) as a benchmark. Define the feasible set as $\mathcal{F}(x; \gamma, l) = \{y \in \mathbb{R} | y \geq (1-\gamma)x - l\}$. The planner solves the following problem:

$$V(x) = \max_{y \in \mathcal{F}(x;\gamma,l)} \left\{ \Gamma(x, y; \alpha, \gamma) + \delta V(y) \right\} \tag{3}$$

[16] Devaney (1989) originally included sensitive dependence on initial conditions in the definition of topological chaos; BBCD+ (1992) subsequently proved that it is implied by transitivity and a dense orbit. Indeed, for continuous maps on an interval (the class of maps of interest in the analysis of this paper), Vellekoop and Berglund (1994) proved that transitivity also implies a dense set of periodic points, so that topological transitivity is the essential property defining chaos.

where $\Gamma(x, y; \alpha, \gamma) = (1+\alpha)K \cdot (y - (1-\gamma)x) - 2e(x)$ and $V(y)$ is the planner's continuation value function at y. Note that $\Gamma(x, y; \alpha, \gamma)$ is continuous, differentiable in y for a given x, and concave in x, y, strictly with respect to x alone. By a standard argument, we can show that there exists a unique V^* satisfying (3); and that this V^* is strictly concave and differentiable. In the quasilinear environment with quadratic $e(\cdot)$ described above, the optimal policy Y^* that solves (3) is also uniquely defined and admits a unique steady state: $x^{**} = \widehat{x} + [(1+\alpha)/(2\beta)] \cdot (1/\delta - (1-\gamma))K$. The state variable monotonically converges to x^{**} for any initial condition x_0.

3. Political equilibrium

3.1 *Existence of equilibria with no attracting steady state*

We now turn to the study of the equilibria of the game in which policies are chosen by the incumbent party (henceforth, the incumbent) under the assumption of quasilinear preferences (1). The goal of this subsection is to prove the existence of equilibria with cycles or aperiodic dynamics. The exact type of dynamics that is possible is studied in the next subsection.

The incumbent's problem can be written as follows:

$$\max_{y \geq (1-\gamma)x - l} \left\{K\left[y - (1-\gamma)x\right] - e(x) + \delta v(y)\right\}. \tag{4}$$

The incumbent maximizes the expected utility of her constituency taking the expected continuation value v as given, thus ignoring the cost/benefit for the constituency of the other party, $\alpha K[y - (1-\gamma)x] - e(x)$. In equilibrium, the expected continuation in state x, the value function v must satisfy

$$\begin{aligned} v(x) = {} & \frac{1}{2}\left[K\left(y(x) - (1-\gamma)x\right) + \delta v\left(y(x)\right)\right] \\ & + \frac{1}{2}\left[\alpha K\left(y(x) - (1-\gamma)x\right) + \delta v\left(y(x)\right)\right] - e(x). \end{aligned} \tag{5}$$

If x is the state, each party suffers a disutility $e(x)$ for sure; with probability $1/2$ the incumbent remains in office and selects $y(x)$, obtaining $K(y(x) - (1-\gamma)x) + \delta v(y(x))$; with probability $1/2$ the party is no longer in office and receives only $\alpha K(y(x) - (1 - \gamma)x) + \delta v(y(x))$, since the policy $y(x)$ is selected by the other party. An equilibrium is characterized by a pair of functions $y(\cdot)$ and $v(\cdot)$ such that for all states x, $y(\cdot)$ solves (4) given $v(\cdot)$ and $v(\cdot)$ solves (5) given $y(\cdot)$.

The incumbent's trade-off can be described as follows. By increasing y, s/he increases current utility for his/her district; by increasing y, however, s/he also affects future's utility for all through the effects on the expected continuation function v evaluated at y. There are two key differences between (4)–(5) and the planner's problem (3). The first is that, as we mentioned above, in any given period the incumbent selects a policy that maximizes the expected utility of his/her constituency alone, ignoring the spillover effects on the constituency of the other party. The second (and most important) difference is that the value of the incumbent's problem (4) does not coincide with the incumbent's continuation value function (5) except in the special case in which $\alpha = 1$. The value of (4) is the expected value for the incumbent; in the continuation of the game, however, the incumbent at t will remain incumbent only with probability $1/2$. This feature make the incumbent's problem time inconsistent since her objective function when selecting the policy does not coincide with the expected continuation value.

In the planner's solution, the marginal effect of the state on the expected continuation value is independent of expected future policy y:

$$\begin{aligned}\left[V^*\right]'(x) &= -(1+\alpha)(1-\gamma)K - 2e'(x) + \left[(1+\alpha)K + \delta\left[V^*\right]'\left(Y^*(x)\right)\right]\left[Y^*\right]'(x) \\ &= -(1+\alpha)(1-\gamma)K - 2e'(x) \qquad (6)\end{aligned}$$

where $[Y^*]'$ is the derivative of the planner's policy function and the second equality follows from the envelope theorem.

In the political equilibrium, however, the standard envelope theorem is not directly applicable, making the optimal decision for the incumbent critically dependent on her expectation of future behavior of the other party. The incumbent's value function (5) can be written as

$$\begin{aligned}v(x) = &\left[K\left(y(x) - (1-\gamma)x\right) + \delta v\left(y(x)\right)\right] - e(x) \\ &- \frac{1}{2}(1-\alpha)K \cdot \left[y(x) - (1-\gamma)x\right] \qquad (7)\end{aligned}$$

where the first line on the right-hand side is the objective function that is maximized by the incumbent at $t+1$, and the second line collects the wedge between the incumbent's objective function and the expected continuation value. Applying the envelope theorem to the first line of (7),[17] we have

$$v'(x) = -e'(x) - (1+\alpha)K(1-\gamma)/2 - (1-\alpha)Ky'(x)/2. \qquad (8)$$

The key feature of this expression is that the marginal change in the value function depends on the expected policies selected by future incumbents, i.e., $y(\cdot)$. If the incumbent at t expects the incumbent in the following period (herself or the opponent) to rapidly increase the policy as a function of the state (i.e., a high positive $y'(\cdot)$), then s/he will have higher incentives to keep the state low; similarly, if s/he expects the future incumbent to reduce the state or to increase it slowly (i.e., a low or negative $y'(\cdot)$), then s/he will have higher incentives to increase the state. The important question for predicting behavior in a political equilibrium is what kind of expectations on $y(\cdot)$ are consistent with equilibrium behavior.

In equilibrium, the policy must solve (4). At a state x that satisfies the first-order condition for optimality in (4), we therefore must have $K = -\delta v'(x)$. Ignoring the policy constraint for the moment, an interior equilibrium satisfies both this condition and (8). Combining these two conditions, we obtain

$$K/\delta = e'(x) + (1+\alpha)K(1-\gamma)/2 + (1-\alpha)Ky'(x)/2. \qquad (9)$$

[17] The assumption of differentiability here is without loss of generality, since as we will show in Proposition 1, the equilibrium is almost everywhere differentiable (and always in the relevant region). We assume here differentiability only for ease of notation, since the same argument can be made without assuming differentiability.

This is a simple differential equation that can be solved in closed form up to a free constant c. The solution can be written as

$$\Psi(x,c)=\frac{2/\delta-(1+\alpha)(1-\gamma)}{1-\alpha}\cdot x-\frac{2}{K(1-\alpha)}e(x)+c \tag{10}$$

Under assumption (2), a functional form that we will assume for the rest of this section and in Section 4, the solution of (9) becomes

$$\psi(x,c)=\varphi_1\cdot x-\varphi_2\cdot x^2+c, \tag{11}$$

where

$$\varphi_1=\frac{2/\delta-(1+\alpha)(1-\gamma)+2\beta\widehat{x}/K}{1-\alpha},\quad \text{and}\quad \varphi_2=\frac{\beta}{(1-\alpha)K}$$

The following Proposition 1 characterizes a sufficient condition such that an equilibrium exists in which the investment function coincides with (11) in all periods, except for at most a finite transition period where the policy must accommodate the feasibility constraint.

Figure 1 illustrates the equilibrium construction. The figure shows $\psi(x,c)$ (the dashed curve) and its relationship with the full equilibrium investment function (the solid curve $y^*(x,c)$, partly overlapping with $\psi(x,c)$, formally described below). The quadratic curve described by $\psi(x,c)$ reaches a maximum $\overline{x}(c)=\psi(x^*,c)$ at $x^*=\varphi_1/(2\varphi_2)$, and has two fixed points at $x^*_-(c)$ and $x^*_+(c)$. If $\psi(\cdot,c)$ were the equilibrium investment function for the entire domain of x, then the policy would increase the state variable for states in $(x^*_-(c),x^*_+(c))$, since in this region $\psi(x,c)>x$; and it would decrease the state variable for states $x<x^*_-(c)$ and $x>x^*_+(c)$, since in this region $\psi(x,c)<x$. For high values of x above $x^*_+(c)$, or low values below $x^*_-(c)$, however, we may have $\psi(x,c)<(1-\gamma)x-l$; a violation of the feasibility constraint. To avoid this, the equilibrium construction of Proposition 1 truncates $\psi(\cdot,c)$ on the left, setting it equal to $\psi(x^*_-(c),c)$ for $x\le x^*_-(c)$; and, on the right, by setting it equal to $\max\{\psi(x^*_+(c),c),(1-\gamma)x-l\}$ for $x\ge x^*_+(c)$. This gives us an "adjusted" investment function:

$$y^*(x,c)=\begin{cases}\max\{\psi(\underline{x}(c),c),\psi(x,c)\} & x\le\overline{x}(c)\\ \max\{\psi(\overline{x}(c),c),(1-\gamma)x-l\} & x>\overline{x}(c),\end{cases} \tag{12}$$

where $\underline{x}(c)=x^*_-(c)$ and $\overline{x}(c)=\psi(\varphi_1/(2\varphi_2),c)$. Proposition 1 proves that, for an appropriately chosen set $\mathcal{C}^*$ of integration constants c, (12) is an equilibrium investment function with no attracting steady states.

The proof has three steps. First, we show that, for an appropriate choice of c, $\psi(\cdot,c)$ does not have attracting steady states, and it is such that once the state enters the set $[\underline{x}(c),\overline{x}(c)]$, it never exits. Using this result, we then find conditions under which (12) does not violate the feasibility constraint, it admits no attracting steady state, and it is such that the state variable enters the set $[\underline{x}(c),\overline{x}(c)]$ in finite time and never exits it. Finally, and most importantly, we prove that if players expect (12) as an investment function, then they find it optimal to invest according to (12).

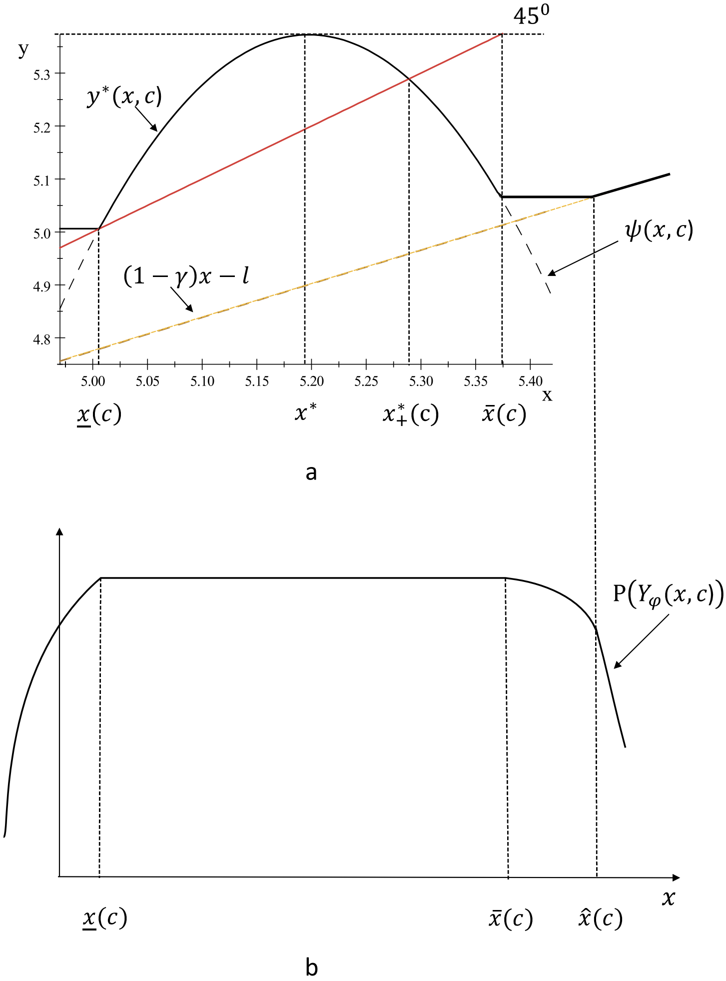

FIGURE 1. The equilibrium construction.

Lets us start from the first property. A key feature of $\psi(x, c)$, as illustrated in panel 1.a of Figure 1, is that for any $x \in [\underline{x}(c), \overline{x}(c)]$ we have $\psi(x, c) \in [\underline{x}(c), \overline{x}(c)]$, and the derivative $\psi'(x, c)$ of $\psi(x, c)$ with respect to x is larger than 1 in absolute value at any fixed point in $[\underline{x}(c), \overline{x}(c)]$, so the policy is repelled by them. These properties, however, depend on the choice of the constant of integration c. As we shift c up or down, we shift $\psi(\cdot, c)$ up or down; and we thus change the dynamics associated to it. On the one hand, the parameter c cannot be selected too small, otherwise $x^*_+(c)$ would be too close to x^*, and the absolute value of the derivative at $x^*_+(c)$ would be lower than 1, thus making $x^*_+(c)$ an attracting steady state. On the other hand, c cannot be selected too high: $\overline{x} = \psi(x^*, c)$ would be too high, $\psi(\overline{x}, c)$ too small, and $\psi(\overline{x}, c) < \underline{x}(c)$; so, $\psi(x, c)$ would not be a self-map in $[\underline{x}(c), \overline{x}(c)]$.

The following lemma characterizes the exact conditions on c for $\psi(x, c)$ to be a self-map in $[\underline{x}(c), \overline{x}(c)]$ with no attracting steady state. To state it, define the following correspondence: $X^*(c) = [\underline{x}(c), \overline{x}(c)]$, where the thresholds $\underline{x}(c)$ and $\overline{x}(c)$ are defined as above. Under the assumption that $e(x)$ is quadratic as in (2), an assumption that is maintained in this and the next sections, we have

$$\underline{x}(c) = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c}}{2\varphi_2}, \quad \text{and} \quad \overline{x}(c) = \frac{\varphi_1^2 + 4\varphi_2 c}{4\varphi_2}$$

We can now state the following.

Lemma 1. *Let $c \in \mathcal{C}^*$, where $\mathcal{C}^*$ is defined as*

$$\mathcal{C}^* = \left[\frac{(3 - \varphi_1)(1 + \varphi_1)}{4\varphi_2}, (4 - \varphi_1)(2 + \varphi_1)/(4\varphi_2)\right] \tag{13}$$

Then $\psi(\cdot, c)$ has no attracting steady state and $\psi(x, c) \in X^*(c)$ for any $x \in X^*(c)$.

Proof. See Section A.1 in the Appendix. □

We will now use Lemma 1 to find conditions under which $y^*(\cdot, c)$ is feasible and admits no attracting steady state. The policy function is feasible if $y^*(x, c) \geq (1 - \gamma)x - l$ for all $x \in \mathbb{R}$. By construction, it is certainly the case that $y^*(x, c)$ is feasible for all $\gamma \in (0, 1)$, $l > 0$, and $x \leq x^*_-(c)$, since we must have $(1 - \gamma)x^*_-(c) - l < x^*_-(c)$. We might however have that $(1 - \gamma)\overline{x}(c) - l > y^*(\overline{x}(c), c)$, which would imply $y^*(x, c)$ violates feasibility in a neighborhood of $\overline{x}(c)$. Naturally, we can always find thresholds on γ and l such that feasibility is satisfied. We can however characterize a more general condition on the preferences of the players and the technology such that feasibility is satisfied for any γ and l. Define R to be the ratio $R{=}\beta/K$. This ratio captures the temptation for an incumbent to abuse its position in selecting the policy. The numerator measures the importance of the externality generated by x on society; the denominator measures the importance of the private benefit of the policy for the incumbent. To state Lemma 2 below, define the threshold:

$$R^*(\alpha) = \frac{4\delta(1 - \alpha)(2 - \gamma) + \delta(1 + \alpha)(1 - \gamma)\gamma - 2\gamma}{2\delta(\widehat{x}\gamma + l)} \tag{14}$$

We have the following.

Lemma 2. *If $R \geq R^*(\alpha)$ and $c \in \mathcal{C}^*$, then the investment function $y^*(\cdot, c)$ as defined in (12) is feasible, it has no attracting steady states, and it is such that the state variable enters the set $X^*(c)$ in finite time and never leaves it.*

Proof. See Section A.2 in the Appendix. □

We will study in detail in the next section what type of dynamics is impressed by $y^*(x, c)$ on the state x, i.e., the type of cycles that are feasible in equilibrium and whether chaotic trajectories exist. Before turning to that, we complete here the equilibrium characterization by proving that $y^*(x, c)$ is indeed a best response, and thus an equilibrium. This is done in the proof of the following proposition.

Proposition 1. *Consider an economy with $R \geq R^*(\alpha)$:*

- *For any $c \in C^*$, $y^*(x, c)$ is an equilibrium policy function with no attracting steady state.*
- *In this equilibrium, the state variable is in $X^*(c)$ for all periods except at most for a finite transition period.*

Proof. We have already proven in Lemmata 1–2 that $y^*(\cdot, c)$ is a feasible self-map in $[\underline{x}(c), \overline{x}(c)]$ that admits no attracting steady state. We now prove that if the players expect $y^*(\cdot, c)$ to be the equilibrium policy used by future policy makers, then they find it optimal to invest according to $y^*(\cdot, c)$. A policy maker chooses a policy θ that solves (4), and thus maximizes the objective function: $P : \theta \to K\theta + \delta v(\theta)$. We will show that P is almost everywhere differentiable (and indeed differentiable in $(\underline{x}(c), \overline{x}(c))$), concave in $\mathbb{R}$, and maximal in $[\underline{x}(c), \overline{x}(c)]$. This implies that that for any x, it is optimal to choose a point $y^*(x, c) \in [\underline{x}(c), \overline{x}(c)]$. We proceed in 3 steps.

Step 1. Consider first policies $\theta \in [\underline{x}(c), \overline{x}(c)]$. In this region, we have $y^*(\cdot, c) = \psi(\cdot, c)$, as defined in (9). By the definition of $\psi(\cdot, c)$, we therefore have $v'(\cdot) = -K/\delta$, so P is constant and differentiable in this interval. In $(\underline{x}(c), \overline{x}(c))$, the derivative of the objective function is such that

$$P'(\theta) = K + \delta v'(\theta) = K - \delta\left[e'(\theta) + \frac{(1+\alpha)K(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}\psi'(\theta, c)\right] = 0, \tag{15}$$

where in the last equality we used (9).

Step 2. Consider now policies $\theta < \underline{x}(c)$ and $\theta > \overline{x}(c)$. In this region, $P(\cdot)$ is also obviously differentiable, since the policy if constant in this region. Assume first $\theta > \overline{x}(c)$. Note that

$$\overline{x}(c) = \psi\left(\frac{\varphi_1}{2\varphi_2}, c\right) \geq \frac{1}{4\varphi_2}\left[\varphi_1^2 + 1 - (\varphi_1 - 1)^2\right] = \frac{\varphi_1}{2\varphi_2} = x^*$$

where $x^* = \arg\max_z \psi(z, c)$. Since $\overline{x}(c) \geq \varphi_1/(2\varphi_2)$, $\psi(\cdot, c)$ is concave, and $\psi'(\varphi_1/(2\varphi_2), c) = 0$, we conclude that $\psi'(\theta, c) \leq 0$. We therefore have

$$\begin{aligned} P'(\theta) &\leq K - \delta\left[e'(\theta) + \frac{(1+\alpha)K(1-\gamma)}{2}\right] \\ &\leq K - \delta\left[e'(\theta) + \frac{(1-\alpha)K}{2}\big((1-\gamma) + \psi'(\theta, c)\big)\right] = 0 \end{aligned} \tag{16}$$

for any $\theta > \overline{x}(c)$. Consider now $x < \overline{x}(c)$. Naturally, $\underline{x}(c) < \varphi_1/(2\varphi_2)$, so

$$P'(\theta) = K - \delta\left[e'(\theta) + \frac{(1+\alpha)K(1-\gamma)}{2}\right] \geq K - \delta\left[\begin{array}{c} e'(\theta) \\ + \frac{(1-\alpha)K}{2}\big((1-\gamma) + \psi'(\theta, c)\big) \end{array}\right] = 0. \tag{17}$$

Step 3. Conditions (16) and (17) imply that $P(\cdot)$ achieves a maximum at any point in $[\underline{x}(c), \overline{x}(c)]$. To see the concavity of $P(\cdot)$, note that it is continuous, concave with positive derivative in $\theta < \underline{x}(c)$, flat in $\theta \in [\underline{x}(c), \overline{x}(c)]$, and concave with negative derivative in $\theta > \overline{x}(c)$. To see that $y^*(x, c)$ is an optimal policy for the incumbent in state x note that by (15)–(17), $P(\cdot)$ achieves a maximum in $[\underline{x}(c), \overline{x}(c)]$ and that, when $c \in \mathcal{C}^*$, $y^*(x, c) \in [\underline{x}(c), \overline{x}(c)]$ for any x for which it is feasible; and for a state x in correspondence of which no θ in $[\underline{x}(c), \overline{x}(c)]$ is feasible, then the policy is at a constrained optimum. □

Figure 1 illustrates $y^*(\cdot, c)$ and its relationship with the objective function in (4). In $x \in [\underline{x}(c), \overline{x}(c)]$, $y^*(x, c)$ is equal to $\psi(x, c)$ and this function maps $[\underline{x}(c), \overline{x}(c)]$ to itself. Because of this, the players expect that the future dynamics is driven by $\psi(\cdot, c)$. By its definition, this function keeps the expected utility of a policy maker (i.e., the objective function in (4)) constant and at its maximal level. Because $y^*(x, c) \in [\underline{x}(c), \overline{x}(c)]$ for any $x \in \mathbb{R}$, not just $x \in [\underline{x}(c), \overline{x}(c)]$, the policy is optimal. The players are indifferent between all policies chosen in equilibrium in any state just as in a mixed equilibrium, but the investment function is uniquely defined up to the constant c since the derivative of $\psi(\cdot, c)$ (and thus $y^*(\cdot, c)$) must be such that the policy makers are indifferent in $[\underline{x}(c), \overline{x}(c)]$.

We conclude this section with five remarks on this equilibrium. Readers interested in the characterization of equilibrium dynamics may skip the rest of this section on a first read and proceed directly to Section 3.2.

Why is the investment function hump shaped? A key feature of $y^*(\cdot, c)$, which allows it to be a self-map and, as we will see, to generate cycles and chaotic behavior is that it is nonmonotonic and hump-shaped. This shape endogenously emerges from the equilibrium condition (9) discussed above when $e(\cdot)$ is a convex function of x. Recall that $\psi(\cdot, c)$ is chosen so that an incumbent politician in state x_t is indifferent when choosing different values of x_{t+1}. An increase in x_{t+1} generates a constant marginal benefit K, and an increasing marginal cost $e'(x_{t+1})$, for the policy maker. To make the policy maker at t indifferent between different levels of x_{t+1}, these effects must be compensated in equilibrium. This is achieved by a strategy where future policy makers react to the increase in x_{t+1} by reducing the marginal rate of increase in the state. In the pollution example, future policy makers move from strategic complements for low states (when they respond to increases in pollution with increases), to strategic substitutes for high states (when they respond to increase in pollution with reductions). In the equilibria of Proposition 1, this induces hump-shaped investment functions in which the rate of investment $[y^*]'(\cdot, c)$ is declining, first positive and then negative as in Figure 2.

Multiplicity and the role of the constant of integration **c**. Proposition 1 characterizes the set of values of c for which it is possible to construct an equilibrium in which the dynamics is driven by the solution of (11), except for at most a finite transition period. As mentioned, these are equilibria analogous to mixed-strategy equilibria: the strategy is optimal because policy makers at $t+1$ choose policies in $[\underline{x}(c), \overline{x}(c)]$ according to a strategy that makes the policy maker at t indifferent between policies in $[\underline{x}(c), \overline{x}(c)]$. There is a continuum of equilibria in the model because the condition under which policy makers are indifferent between values of x in $[\underline{x}(c), \overline{x}(c)]$ defines the investment function only up to a constant. The set of admissible constants is restricted by other equilibrium conditions, but it is still a nonempty compact set.

The value of the constant of integration c is important because it determines the slope of the solution ψ, and thus of the investment function, in correspondence to its fixed point $x^*_+(c)$. Consider the example illustrated in Figure 1, in which $\psi(x^*, c) > x^*$. If we choose a very small c, we have $x^* \simeq \psi(x^*, c)$: so, $x^* \simeq x^*_+(c)$, implying $\psi'(x^*_+(c), c) \simeq 0 > -1$. In this case, $x^*_+(c)$ is an attracting steady state, and we cannot have stable cycles or chaos. For larger values of c, Lemma 1 shows that we can have $\psi'(x^*_+(c), c) < -1$. In this case, $x^*_+(c)$ is repelling and we cannot have a stable steady state in $X^*(c)$; the equilibrium must generate a stable cycle or chaos.

In terms of economic interpretation, a higher value of c, corresponds to situations in which, in a neighborhood of the steady state $x^*_+(c)$, the policy maker at $t+1$ responds to a marginal increase in the state x_t at t by a larger decrease in x_{t+1}. If we interpret x_t as pure pollution therefore equilibria with larger c correspond to situations in which in a neighborhood of the steady state $x^*_+(c)$ policy makers respond to a marginal increase in pollution by reducing pollution more aggressively, thus cleaning up more aggressively "the mess" inherited by the previous generation of policy makers. Note that the marginal response of policy makers at $x^*_+(c)$ is important because it determines how strongly the state is repelled from $x^*_+(c)$: depending on how strongly it is repelled (and, therefore, on c), we may have stable cycles or chaotic trajectories.

How strong is the existence condition? A number of parameters in the model contribute to making it easier or more difficult to have equilibria with a nonconverging orbit. For example, it is clear that if both $\gamma = 0$ and l is arbitrarily close to 0, then it is impossible to construct cycles or nonconverging orbits. The reason is that in this case the policy constraint $x^{t+1} \geq (1-\gamma)x^t - l$ forces the policy to be monotonically increasing over time, since $x^{t+1} = y(x^t) \geq x^t$ for $\gamma = 0$, $l \to 0$. And indeed, consistently with this observation, we have that $R^*(\alpha) \to \infty$ as both $\gamma \to 0$ and $l \to 0$. Remarkably, however, cycles and nonconverging orbits exist even for arbitrarily small (but strictly positive) values of γ and l, if we choose the other parameters δ, $\widehat{x}$, and α appropriately. For example, it can be verified that $R^*(1) < 0$, so the equilibrium of Proposition 1 always exists when α is sufficiently close to 1, i.e., when time inconsistency is not too large (as we will discuss more extensively in Section 4.1). Two other important variables are the discount factor δ and the ideal point for society $\widehat{x}$. The threshold $R^*(\alpha)$ is increasing in δ, so the smaller is the discount factor the easier is to satisfy the sufficient condition in Proposition 1. It is indeed interesting to note that a small enough discount factor is sufficient for the existence of nonconverging equilibria. A small discount factor however is not necessary, and the sufficient condition can be satisfied for any δ. On the contrary, the threshold $R^*(\alpha)$ is decreasing in $\widehat{x}$, so a larger ideal point makes nonconverging equilibria easier to achieve. Nonconverging orbits are however possible even if $\widehat{x} = 0$. The most interesting variable in $R^*(\alpha)$ is α, which measures the degree of time inconsistency in the economy. We postpone the discussion of the relationship between time consistency and nonconverging equilibria to Section 4.1. In the model of Section 2, we have assumed that $\widehat{x} \geq 0$ and $l > 0$, although possibly both variables may be arbitrarily small.

The result of Proposition 1 does not require these natural but simplifying assumptions. The general sufficient condition for Proposition 1 is (32) in Section A.2, where Lemma 2 is proven. If we assume $\widehat{x} < 0$ and/or $l \geq 0$, but $\widehat{x}\gamma + l \geq 0$, (32) continues to imply (33), so the analysis remains completely unchanged (with the minor convention of defining $R^*(\alpha) = \infty$ if $\widehat{x}\gamma + l = 0$). If $\widehat{x} < 0$ and/or $l \geq 0$, but $\widehat{x}\gamma + l < 0$, instead (32) requires $R \leq R^*(\alpha)$. This means that when the ideal point $\widehat{x}$ is negative and sufficiently large in absolute value, then the feasibility condition in Proposition 1 holds if R is sufficiently small.

Unstable steady states. In the equilibria characterized in Proposition 1, the state variable never diverges to $\pm\infty$, since $y^*(x, c)$ is bounded for any $x_0 \in \mathbb{R}$ (see (12)). The equilibria, however, besides the stable cycles and nonperiodic trajectory that we will study in greater detail in the next section, admit unstable steady states. For instance, consider the points $x^*_+(c)$ and $x^*_-(c) = \underline{x}(c)$ in Figure 1, where $\psi(x, c)$ intersects the 45^o line. These points are unstable because, for any open neighborhood U containing them, there exist points within U from which the trajectory moves away from the steady state. The first is unstable for perturbations that move the state both to the right or left, while the second is unstable for perturbations that move the state to the right. Starting from any generic initial state $x_0 > x^*_-(c)$ different from $x^*_+(c)$, the equilibrium investment function pushes the state away from $x^*_+(c)$ and $\underline{x}(c)$, making them "invisible." The unstable steady state $x^*_+(c)$ is inevitable and present in any equilibrium; however, the second, $\underline{x}(c)$, can be eliminated in almost all parametrizations with a minor adjustment to the equilibrium construction. To see this, note that to prevent the state from diverging to $-\infty$ in (12) for $x_0 < \underline{x}(c)$, we require the investment function to remain constant and equal to $\psi(\underline{x}(c)), c)$ for states to the right of $\underline{x}(c)$, creating a kink that "touches" the 45° line from above (see, for instance, Figure 1). However, this is unnecessary for any $c \in \mathcal{C}^{**}$, where

$$\mathcal{C}^{**} = \left(\frac{(3-\varphi_1)(1+\varphi_1)}{4\varphi_2}, \frac{(4-\varphi_1)(2+\varphi_1)}{4\varphi_2}\right].$$

For $c \in \mathcal{C}^{**}$, we have $\psi(x, c) \in (\underline{x}(c), \overline{x}(c)]$ for any $x \in (\underline{x}(c), \overline{x}(c)]$, so that we have $y^*(x, c) \in (\underline{x}(c), \overline{x}(c)]$ for all $x \in \mathbb{R}$ if we define $y^*(x, c) = \max\{\psi(\underline{x}(c)) + \epsilon, c), \psi(x, c)\}$ when $x \leq \overline{x}(c)$, for some $\epsilon > 0$ sufficiently small. Let $x(c, \epsilon)$ the point such that $\psi(x(c, \epsilon), c) = \psi(\underline{x}(c)) + \epsilon$. With this modification, the investment function is flat and equal to $\psi(\underline{x}(c)) + \epsilon$ for $x \leq x(c, \epsilon)$, and equal to (12) for $x > x(c, \epsilon)$; that is, it is just like (12), but it is flattened at a marginally higher value for low values of x. The resulting investment function intersects the 45° line only at the unstable steady state $x^*_+(c)$: starting from any $x_0 \in \mathbb{R}$, the state enters the set $(\underline{x}(c), \overline{x}(c)]$ and never exits. It is straightforward to verify that for sufficiently small ϵ (which indeed can be made arbitrarily small), the modified investment function satisfies the feasibility constraints and remains a best response, as proven in Proposition 1. While the change described above slightly complicates the formula for $y^*(x, c)$, restricting c to $\mathcal{C}^{**}$ does not qualitatively alter the results, as the equilibria generated with $c \in \mathcal{C}^{**}$ are sufficient to produce stable cycles with more than two periods and chaotic behavior.

An upper bound on investment. In the preceding analysis, we assumed a lower bound in investments so that $y \geq (1-\gamma)x - l$ for $\gamma \in (0,1)$ and $l > 0$, but no upper bound. Bounds on investments are implicit in the model because the marginal cost of large investments is increasing in x and diverges to $-\infty$ as $x \to \infty$. The assumption of no explicit upper bound, however, clearly helps to simplify the presentation; we could have instead assumed, for example, an upper bound $y \leq \Omega_1 x + \Omega_2$ for some $\Omega_1 > 1$ and $\Omega_2 > 0$. In Figure 1, this would appear as a positively sloped line above the 45^o line. For Ω_1, Ω_2 sufficiently large, the constraint would not be binding, but it may be depending on the parametrization of the model.

3.2 *Characterization of the dynamics*

Proposition 1 does not specify whether the equilibrium dynamics is cyclical, and if it is cyclical, the period of the orbit. When the equilibrium orbit converges to a stable cycle, the equilibrium is inefficient, but it is predictable since the orbit follows a well-defined deterministic path. Unpredictability becomes a problem only when the orbit is aperiodic and chaotic (as defined in Section 2). What kind of dynamics can we generate as we vary c in the set $\mathcal{C}^*$?

To address this question, it is useful to "rescale" (12) by an homeomorphism h.[18] Let us denote the composition of two functions by $f \circ g(\cdot) = f(g(\cdot))$.

Definition 1. Let $f : Z_1 \to Z_1$ and $g : Z_2 \to Z_2$ be two maps, we say that f and g are topologically conjugate if there exist a homeomorphism $h : Z_1 \to Z_2$ such that $h \circ f = g \circ h$.

It is important to establish whether two functions f and g are topologically conjugate, because topologically conjugate functions have the same dynamical properties. We have that $[f]^n = [h^{-1} \circ g \circ h]^n = h^{-1} \circ g^n \circ h$, so if x is a fixed point of $[f]^n$, then h must be a fixed point of $[g]^n$, since we have $[g]^n \circ h(\cdot) = h \circ [f]^n(\cdot) = h(\cdot)$. Indeed, the function h gives a one-to-one correspondence between the periodic points of f and g. Periodic and aperiodic orbits of f are mapped by h into qualitatively similar orbits of g. Moreover, f is topologically chaotic (following Devaney (1989, Chapter 1.7)) and admits an absolutely continuous ergodic distribution if and only if the same is true for g. We can therefore study the properties of f by studying g.

An adequate rescaling of (11) by an homeomorphism simplifies the analysis of the equilibria of Proposition 1 because it allows us to link equilibrium dynamics to the dynamics of the logistic function $L_\eta : x \to \eta x(1-x)$, one of the few nonlinear functions for which the dynamics has been extensively studied (see, for instance, Ulam and von Neumann (1947), Ruelle (1977), Jakobson (1981)). Naturally, an equilibrium $y^*(\cdot, c)$ will never be conjugate to the logistic L_η on the entire real line, since L_η is an unbounded function while the equilibrium must satisfy the feasibility constraint $y \geq (1-\gamma)x - l$. To characterize the equilibrium dynamics, however, it is sufficient to have conjugacy on a superset of the support of the states reached in equilibrium. We have the following.

[18] A function between two topological spaces I and J, $g : I \to J$, is said to be a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse g^{-1} is also continuous.

Lemma 3. *Assume $R \geq R^*(\alpha)$ as defined in Proposition 1. For any $\eta \in [3, 4]$, there is a constant*

$$c(\eta; \varphi_1, \varphi_2) = \frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - (\eta/2)(1-\eta/2)\right] \tag{18}$$

such that the equilibrium investment function $y^*(\cdot, c(\eta; \varphi_1, \varphi_2))$ on $X^*(c(\eta; \varphi_1, \varphi_2))$ is topologically conjugate to L_η on $[0, L_\eta(1/2)]$.

Proof. See Section A.3 in the Appendix. □

Consider an equilibrium with constant $c \in \mathcal{C}^*$. By Proposition 1, once the state x enters $X^*(c)$, it never exits. But what are the characteristics of the dynamic behavior in this set? Lemma 3 allows to characterize its dynamics. It tells us that there is a constant $\eta = c^{-1}(c; \varphi_1, \varphi_2)$ such that $y^*(\cdot, c)$ is conjugate in $X^*(c)$ to L_η in $[0, L_\eta(1/2)]$.[19] It follows that the dynamics of $y^*(\cdot, c)$ is qualitatively equivalent to the dynamics of L_η.

Perhaps more interestingly, Lemma 3 allows to construct equilibria with cycles of various periods. Given any type of dynamics generated by L_η, we can find a constant $c = c(\eta; \varphi_1, \varphi_2)$ such that there is an equilibrium function $y^*(\cdot, c)$ having the same dynamics. It is well known that for values $\eta \in [3, 1+\sqrt{6}]$, the logistic has a unique stable cycle of period 2.[20] We can therefore construct an equilibrium with a cycle of period 2 by setting the constant c^* in the equilibrium $y^*(x, c^*)$ of Proposition 1 at $c^* = c(\widehat{\eta}; \varphi_1, \varphi_2)$ for any $\widehat{\eta} \in [3, 1+\sqrt{6}]$. For example, the equilibrium in the left panel of Figure 2 is constructed by setting c^* in (11) equal to $c(3.3; \varphi_1, \varphi_2)$.[21] As η increases beyond $1+\sqrt{6}$, cycles of order $2m$ for any $m \geq 1$ emerge; and for $\eta > 1+2\sqrt{2}$ there are isolated values of η for which cycles with period 3 appear.[22] The equilibrium with a stable cycle of period 3 in Figure 2 is indeed constructed setting $c^* = c_3 = c(3.839; \varphi_1, \varphi_2)$.[23]

In addition to stable cycles, the literature has also identified specific values of η in $[\eta_\infty \simeq 3.5699, 4]$ for which L_η displays chaotic behavior, for example, $\eta = 4$ (see Ulam and von Neumann (1947)); or the Ruelle's constant η^*, which is approximately 3.6785735 (see Ruelle (1977)).[24] Lemma 3 shows that both $c(4; \varphi_1, \varphi_2)$ or $c(\eta^*; \varphi_1, \varphi_2)$ are in $\mathcal{C}^*$; we

[19]Note that $c(\eta; \varphi_1, \varphi_2)$ is invertible in $\eta \in [3, 4]$. We denote here its inverse $c^{-1}(c; \varphi_1, \varphi_2)$ for $c \in \mathcal{C}^*$.

[20]See Devaney (1989), among others.

[21]Specifically, to construct the left panel of Figure 2, we set the exogenous parameters of the model to $\delta = 0.95$, $\gamma = 0.1$, $\alpha = 0.8$, $x = 1$, and $\beta/K = 2$, which imply $\varphi_1 = 22.876$, $\varphi_2 = 10$, and $\mu = 3.3$. In this case, we obtain $c(3.3; \varphi_1, \varphi_2) = -11.832$. The periodic points of the attracting 2-period cycle are $x_1 = 1.1372$ and $x_2 = 1.2504$.

[22]The existence of stable cycles of order 3 is particularly important because by the Sarkovski theorem they imply the existence of cycles of any order. This has sometimes been equated to the presence of chaos. This is however not a completely legitimate interpretation. The logistic has a unique stable cycle and the dynamics converges to it starting from all points in its support except from a subset of measure zero. The additional cycles are unstable cycles that exist only for initial values in a set of measure zero. These cycles are often referred to as "invisible" since for all practical purposes they are unobserved. The lower bound for the existence of a cycle of period 3 has been proven to be $1+2\sqrt{2}$ by, among others, Bechhoefer (1996).

[23]Specifically, to construct the right panel of Figure 2, we assume that the exogenous parameters of the model (i.e., δ, γ, α, $\widehat{x}$, β, K) are such that $\varphi_1 = 22.876$, $\varphi_2 = 10$ (as in the left panel), and $\mu = 3.839$. In this case, we have $c(3.839; \varphi_1, \varphi_2) = -11.762$. The periodic points of the attracting 3-period cycle are $x_1 = 1.0087$, $x_2 = 1.1382$, and $x_3 = 1.3181$. Note that the critical point x^* that maximizes $\varphi(x, c(3.839; \varphi_1, \varphi_2))$ is not a periodic point, since $x^* = 1.1435$.

[24]Ruelle's constant η^* is the only real solution η^* of $(\eta^* - 2)^2(\eta^* + 2) = 16$.

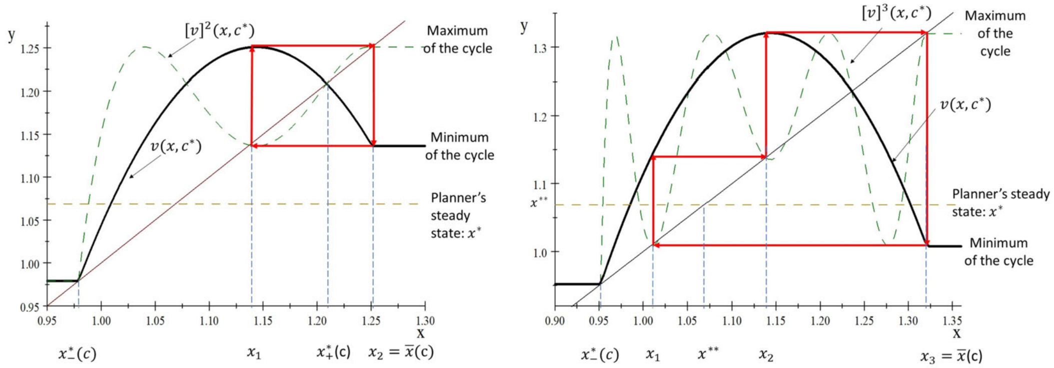

FIGURE 2. One economy, two equilibria with stable cycles of periods $m = 2$ and $m = 3$. The solid (black) line are the investment function $y^*(x, c)$, the dashed (green) line are the iterated maps $[y^*]^2(x, c)$ and $[y^*]^3(x, c)$ for respectively $c(3.3; \varphi_1, \varphi_2)$ and $c(3.839; \varphi_1, \varphi_2)$.

can therefore generate equilibria with the same qualitative properties by setting $c = c_4 = c(4; \varphi_1, \varphi_2)$ or $c = c_{\eta^*} = c(\eta^*; \varphi_1, \varphi_2)$. The top panel of Figure 3 presents the trajectories of two chaotic equilibria with c equal to $c(4; \varphi_1, \varphi_2)$ and $c(\eta^*; \varphi_1, \varphi_2)$, respectively (the parametrizations are always the same as in Figure 2).

The following result is an immediate implication of Proposition 1 and Lemma 3.

PROPOSITION 2. *Assume an economy with $R \geq R^*(\alpha)$ as defined in (14):*

- *For every value $m \geq 2$, there is at least an equilibrium $y^*(x, c_m)$ associated to a point $c_m \in \mathcal{C}^*$ with a unique stable cycle of period at least m. The orbit of this equilibrium is in $[[y^*]^2(x^*, c_m), y^*(x^*, c_m)]$, and its periodic points belong to the set of fixed points of $[y^*]^m(x, c_m)$, as defined in (12).*
- *For values $c \in C^D = (c(\eta_\infty; \varphi_1, \varphi_2), c(4; \varphi_1, \varphi_2)]$ where $\eta_\infty \simeq 3.5699$, the equilibrium $y^*(\cdot, c)$ has an invariant set in which the map is chaotic (topological chaos or ergodic) on $[[y^*]^2(x^*, c), y^*(x^*, c)]$.*

PROOF. See Section A.4 in the Appendix. □

Given Proposition 2, it is also natural to ask what the properties of the long term distribution of states induced by iterations of y^* are. A particularly important property is whether the distribution is absolutely continuous, invariant, and ergodic.[25]

DEFINITION 2. We say that a dynamical system displays ergodic chaos if there is an absolutely continuous probability measure that is ergodic and invariant.

When we have ergodic chaos, the behavior of the dynamical system in the long term can be described by a distribution function. Ulam and von Neumann (1947)

[25]A distribution μ is said to be invariant if $y_*\mu = \mu$, where $y_*\mu$ is the push forward measure $y_*\mu(A) = \mu(y^{-1}(A))$. A distribution is ergodic if $\lim_{T\to\infty} \sum_{k=1}^{T} \varphi([y]^k(x)) = \int \varphi d\mu$ for almost all x.

have famously shown that L_4 admits an ergodic distribution that can be characterized in closed form and is equal to the Arcsine distribution with density function: $\lambda : x \to \pi^{-1}(x(1-x))^{-1/2}$ (see Jakobson (1981)). While this is the only case (apart from topological conjugacy) for which the ergodic distribution of the logistic has been characterized in closed form (and one of the very few dynamical systems for which it can be characterized), subsequent work has shown that there is a set of positive measure of values of η such that L_η admits an ergodic distribution, one of which is Ruelle's number η^* (see Jakobson (1981) and Benedicks and Carleson (1985)).

By Lemma 3, for each of these values, there is an equilibrium of the policy game that qualitatively inherits the same properties. The equilibrium investment function $y^*(x, c_4)$ in $X^*(c)$ with $c_4 = c(4; \varphi_1, \varphi_2)$ has the same properties of the logistic function L_4 on $[0, 1]$. This implies that the distribution generated by the equilibrium correspondent to c_4 can be characterized in closed form, although now its "shape" depends on the fundamentals of the economy. Define the following density $\mu(x; \omega)$ on $X^*(c_4)$:

$$\mu(x;\omega) = \frac{2R}{\pi}\left(16(1-\alpha)^2 - \left(2Rx - \left[\begin{array}{c} \frac{2}{\delta} - (1+\alpha)(1-\gamma) \\ +2R\cdot\widehat{x} \end{array}\right]\right)^2\right)^{-1/2}, \tag{19}$$

where $\omega = (R, \alpha, \delta, \gamma, \widehat{x})$ is the vector of parameters characterizing the economy. We have the following.

Proposition 3. *Assume an economy with $R \geq R^*(\alpha)$. There is a subset of $\mathcal{C}^E \subset \mathcal{C}^D$ with positive measure such that the equilibria $y^*(\cdot, c)$ with $c \in \mathcal{C}^E$ display ergodic chaos in* $[[y^*]^2(x^*, c), y^*(x^*, c)]$*. Among these equilibria, the equilibrium $y^*(x, c_4)$ admits the invariant distribution $\mu(x;\omega)$ in* $[[y^*]^2(x^*, c_4), y^*(x^*, c_4)]$ *defined in (19).*

Proof. See Section A.5 in the Appendix. □

As it can be seen from Figure 3, the density associated to $\mu(x;\omega)$ does not look like the familiar unimodal densities, such as the normal: it has modal values at the extremes, instead than in the interior of its support.[26] This occurs because $y^*(\cdot, c_4)$ is topologically conjugate on $X^*(c)$ to $L_4(\cdot)$ on $[0, 1]$, and the ergodic distribution generated by L_4 is the Arcsine law, which has model values at the extremes, and thus looks like a "U." It should however be noted that the Arcsine law naturally emerges in simple examples. It can be proven that the proportion of time that the one-dimensional Wiener process is positive follows an Arcsine distribution (see Theorem 2 in Feller (1950)), which perhaps suggests that extreme realizations should be less surprising than they are generally expected to be. A nice feature of this example is that despite the equilibrium being chaotic, we can derive a precise relationship between the fundamentals of the model and the ergodic distribution of the state in equilibrium.

[26] Figure 3 has the same parametrization as Figure 2, except that $c(4; \varphi_1, \varphi_2)$ is used to construct the solid trajectory in the top panel and the darker ergodic density in the bottom panel, and $c(\eta^*; \varphi_1, \varphi_2)$ is used for the dashed trajectory in the top panel and the lighter density in the bottom panel.

It is important to note that, if we take the investment function $y^*(x, c^*) = \varphi_1 - \varphi_2 x + c^*$ as an exogenous function of x (i.e., with fixed, exogenous parameters $\varphi = (\varphi_1, \varphi_2)$ and c^*), the existence of the ergodic distribution generated by its dynamics would not be a robust phenomenon, since the set of parameters c^* for which $y^*(x, c^*)$ converges to a stable cycle is open and dense in $[c(1; \varphi_1, \varphi_2), c(4; \varphi_1, \varphi_2)]$.[27] For any $y^*(x, c^*)$ that displays ergodic chaos, we have an arbitrarily close investment function that does not. The cases in which the investment function is exogenous are a relevant benchmark since in the literature the investment function is either assumed as exogenous, or derived in simple models with a unique equilibrium, in which case the function is determined by the parametrization of the environment (i.e., the parameters corresponding to φ_1, φ_2).[28] The key observation here is that the *existence of the equilibrium* with chaos is a robust phenomenon. Proposition 2 shows that if we perturb to the economy (φ_1, φ_2) to a nearby economy $(\widetilde{\varphi}_1, \widetilde{\varphi}_2)$, then there is a “nearby” equilibrium $y^*(x, \widetilde{c}) = \widetilde{\varphi}_1 - \widetilde{\varphi}_2 x + c(c^{-1}(c^*; \varphi_1, \varphi_2); \widetilde{\varphi}_1, \widetilde{\varphi}_2)$ that generates chaotic behavior (here $c^{-1}(\cdot; \varphi_1, \varphi_2)$ is the inverse of $c(\cdot; \varphi_1, \varphi_2)$). The constant c in the equilibrium construction is indeed an *endogenous* variable, not a parameter given by nature such as φ_1, φ_2.[29]

We conclude this section discussing the geometric properties characterizing the investment functions $y^*(\cdot, c)$ for $c \in \mathcal{C}^E$, which generate ergodic chaos. First, note that all equilibrium investment functions $y^*(\cdot, c)$ of Proposition 1, both with and without stable cycles, qualitatively look like the hump shaped functions illustrated in Figure 2. While there is no general characterization of necessary and sufficient conditions on the shape of the investment function y to generate ergodic (or topological) chaos, there is a known general sufficient condition that gives us insights on the geometric features of y that are associated to chaotic behavior.[30] Recall that x^* is the critical point of $y^*(\cdot, c)$, i.e., the point that maximizes $y^*(\cdot, c)$; and $x^*_-(c)$ and $x^*_+(c)$ are the lower and higher fixed points of $y^*(\cdot, c)$, respectively. For these investment functions, $x^* \in (x^*_-(c), y^*(x^*, c))$, so the maximum $y^*(x^*, c)$ is above the 45^o line: for any initial $x_0 \in [x^*_-(c), y^*(x^*, c)]$, the state remains in $[x^*_-(c), y^*(x^*, c)]$ for all iterations. Moreover, we have no stable steady state because the slope of y is larger in absolute value than one at $x^*_-(c)$ and $x^*_+(c)$ (the two unstable steady states). Given this, there are three forces “pushing around” the state. When the state is close to $x^*_+(c)$, it is repelled by it since $[y^*]'(x^*_+(c), c) < -1$. The state can move down below $x^*_+(c)$, or up above $x^*_+(c)$. If the state is pushed down toward $x^*_-(c)$, then it is repelled again to a higher state, so the state must be eventually be pushed up, above $x^*_+(c)$. In this case, however, it eventually reaches a point $x < y^*(x^*, c)$

[27] This follows from the fact that by Lemma 3 $y^*(x, c(\mu; \varphi_1, \varphi_2))$ for $\mu \in [1, 4]$ is topologically conjugated to the logistic with parameter μ, and a fundamental theorem in Graczyk and Świątek (1997) proving that the logistic has an attracting cycle for an open and dense set of parameters μ in $[1, 4]$.

[28] See, for instance, Day (1982) and Benhabib and Day (1982).

[29] To evaluate the comparative statics of the equilibrium dynamics with respect to the change of an exogenous variable, we therefore need a theory of how the players select an equilibrium after the change. A reasonable hypothesis is that after marginal change in the environment, the players would select an equilibrium that leads to similar behavior.

[30] The examples presented in Figure 3 satisfy this condition.

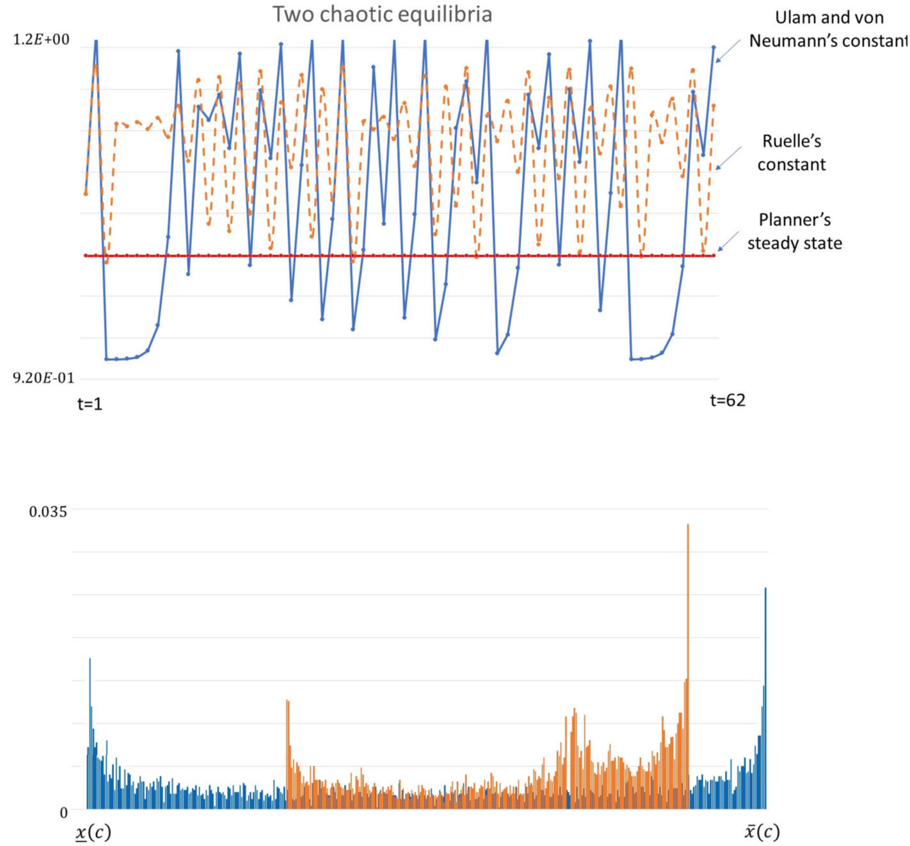

FIGURE 3. Two equilibria with ergodic chaos existing for the same economy. The first equilibrium is topologically conjugate to von Neumann and Ulam's example, the second to Ruelle's example. The top panel shows the state trajectories, and the bottom panel shows the corresponding ergodic distributions.

at which $y^*(x, c) < x$, so it will have to move down. These dynamics may induce the system to converge to a cycle with periodic points in $(\underline{x}^*(c), y^*(x^*, c))$, as in Figure 2. It can however be shown that under regularity conditions satisfied by the equilibria of Proposition 1, a necessary condition for this to occur (and thus for the existence of a stable cycle) is that the orbit originated from the critical point x^* converges to a stable cycle (see Theorem II.4.1 in Collet and Eckmann (1980)). It follows that whenever the orbit starting from the critical point does not converge to a stable cycle, then a stable cycle of any period does not exist. While this does not necessarily imply that we have ergodic chaos, it is indeed the case with sufficient regularity as the equilibria of Proposition 1.[31]

[31] Ergodic chaos requires convergence of the ergodic distribution to an absolutely continuous distribution. A "geometric" sufficient condition is presented by Misiurewicz (1981); it requires that the orbit starting from the critical point x^* enters an unstable cycle. This condition is satisfied by both examples in Figure 3.

3.3 *On the interpretations of the model*

In light of the characterizations of Proposition 1 and 2, it is useful to return on the interpretation of the model. An advantage of its simplicity is that it easily lends itself to alternative interpretations, depending on the choice of its parametrization. Two interpretations stand out. In the first, x is a stock of investments that generates local effects and global externalities. Examples of these investments include investments in highways, airports, or power plants, which are typically at the center of political economic decisions. Power plants have positive effects near their location since they provide cheaper and more reliable electricity, but they may also generate positive or negative externalities to other far away localities (because of, say, pollution). In this interpretation, we may have a positive marginal impact on utility of x for low values, and negative for large values. In this interpretation, the strictly concave function $x - e(\cdot)$ is the instantaneous utility for setting the state at x. This is a completely standard assumption in dynamic public finance models (see, e.g., Battaglini and Coate (2007, 2008) and references therein).[32] Under this interpretation, it is natural to assume $\widehat{x} > 0$. A superficial objection to this interpretation is that the impact of the state enters the utility as a "$-e(x)$," so as a negative value. But this is conceptually irrelevant. We could have assumed that the impact of x was, say, $z_1 + z_2 e(x)$ for $z_1 > 0$, and the analysis would have been analogous. Adding a constant z_1 to the utility function does not really change the preferences in a qualitative way. The feature that is important for the analysis is the marginal effect of x on the utility. The key assumption that we make is that the players utility is strictly concave in x.[33]

An alternative appealing interpretation of the model is that x is just a stock of pollution. In this interpretation, it would be appropriate to assume $e'(x) > 0$ for all $x \in \mathbb{R}$. This interpretation is absolutely consistent with the model. For example, it is consistent with the assumption of a quadratic cost function as (2) with $\widehat{x} = 0$ when the state variable is only positive (so that $e'(x) > 0$ for all $x \in \mathbb{R}$). Examples of this situation can be easily illustrated. Assume $\alpha = 0.9$, $\delta = 0.9$, $\beta/K = 0.1$, $\widehat{x} = 0$, and some $\gamma \in (0, 1)$ and $l > 0$. Suppose, for example, we choose c so that we have 2-period cycle. Despite having $e'(x) > 0$, we have $\varphi_1 = 5.1222$, $\varphi_2 = 1$ and $c = c(3.3, \varphi_1, \varphi_2) = -2.9256$, so, $\psi(x, c) = 5.1222 \cdot x - x^2 - 2.9256$, which is a hump shaped function with an attractive 2-period cycle with periods $x_1 = 3.629$ and $x_2 = 2.4932$. If $R > R(\alpha)$, Proposition 1 shows that we can construct an equilibrium investment function $y^*(\cdot, c)$ that has a unique stable cycle with periods $x_1 = 3.629$ and $x_2 = 2.4932$. Another assumption that would be completely consistent with the assumption of x as pollution is the assumption of an exponential cost function $e(x) = \vartheta_1 \cdot \exp(x - \vartheta_2)$ for some $\vartheta_1, \vartheta_2 > 0$, discussed in Example 1 below. Looking at (10), we can see that it is not necessary for the results that

[32] In Battaglini and Coate (2008), choosing an investment x_t at t generates an investment cost $x_t - (1-\gamma)x_{t-1}$ at t, and a concave utility $u(x_t)$ at $t+1$. This implies that at t the instantaneous utility of an investment x_t is $x_t + \delta u(x_t)$, plus a constant. This corresponds to the model presented here in which $e(x_t) = -\delta u(x_t)$.

[33] Naturally, the specific thresholds characterized in Proposition 1 depend on the functional form in (2), but as we discuss in Section 3.1 they can be generalized to a general class of convex and smooth function $e(x)$.

$e(\cdot)$ has minimum (as, say, in (2) when $\widehat{x} > 0$): to obtain cycles and chaos, we really need $\psi(\cdot, c)$, not $e(\cdot)$, to be "hump shaped" as in the figures. For this property, the assumption that $e(x)$ has a minimum at $\widehat{x}$ (and so that $-e(x)$ has a maximum) is not necessary. Since the first term of $\psi(x, c)$ (i.e., $\varphi_1 \cdot x$) has always positive derivative, $\psi(\cdot, c)$ is hump-shaped even if the "cost function" $e(\cdot)$ is always increasing (which is consistent with the interpretation of the state as pure pollution).

4. The effect of time inconsistency

4.1 *Time inconsistency and the "size" of the chaotic region*

In the model of Sections 2–4, the degree of time inconsistency impressed on the economy by the decision process plays a particularly important role in determining "how much" chaos we can observe in equilibrium. Time inconsistency consists in the discrepancy between objective function maximized by the incumbent selecting the policy (i.e., (4)), and the expected continuation value function at t before the incumbent at $t+1$ is determined (i.e., (7)). In the planner's problem, there is no difference between these two functions. By comparing (4) and (7), we can see that in the political game, instead, the functions differ by

$$-\frac{(1-\alpha)K}{2}\big[y(x) - (1-\gamma)x\big].$$

The parameter $\alpha \in (-\infty, 1]$ captures time inconsistency: as $\alpha \to 1$, time inconsistence converges to zero; as we reduce α, time inconsistency is increased.[34] In this limit case as $\alpha \to 1$, $v'(x)$ is independent of $y'(x)$ and the equilibrium qualitatively looks like the planner's problem. Ignoring the feasibility constraint, the incumbent's optimal policy is

$$y_{nt} = \widehat{x} + \frac{K}{2\beta} \cdot \big(1/\delta - (1-\gamma)\big),$$

and so the equilibrium policy is $y_{nt}(x) = \max\{y_{nt}, (1-\gamma)x - l\}$. This simple dynamical system has a unique steady state y_{nt} that differs from the planner's steady state x^{**} only because it is lower. The effect of α on the "degree of chaos" in the equilibria constructed in the previous sections can be seen from its effect on the size of the chaotic region $\mathcal{K}_\alpha(c) = \|y^*(x^*, c) - [y^*]^2(x^*, c)\|$, i.e., the difference between the maximal and minimal values assumed by the state in the chaotic region. While this region depends on the specific equilibrium (and so on c), it is always the case that it is contained in $X^*(c)$ and, as it is easy to verify, we have $\|X^*(c)\| \leq 4(1-\alpha)K/\beta$. It follows that $\mathcal{K}_\alpha(c) \to 0$ as $\alpha \to 1$. It is therefore the case that, as $\alpha \to 1$ we can still have chaos, but the size of the set in which the state can "wander around" collapses to zero.

Time inconsistency is an important factor to allow complex dynamics in generic environments because it is the reason why we have multiplicity of equilibria in the model. When $\alpha = 1$, (8) shows that the marginal impact of x in equilibrium depends only on the

[34] When $\alpha = 1$, the effect of private expenditure is the same on the constituencies of the party in power and of the other party. The equilibrium policy is still not the utilitarian policy because the incumbent does not internalize the negative externality of x; but it is time consistent.

fundamentals of the model (just as in the planner's case): we cannot have multiple equilibria in this case, since investment is uniquely defined by exogenous marginal cost and benefits. It is only when $\alpha < 1$ that the marginal impact of x on the policy maker's utility also depends on the expected policy $y(\cdot)$ adopted by future policy makers: different policy can therefore be self-generated by the expectations that they generate. Thanks to the multiplicity of equilibria, for any parametrization such that $R > R^*(\alpha)$, we can find a $c \in C^D$ (i.e., an equilibrium) such that complex dynamics occurs.

4.2 *Alternative economic models with time inconsistency*

In the previous analysis, we have assumed a stylized dynamic political economy model in which two parties alternate in power. In this section, we show that the logic behind the type of equilibria with cycles and chaotic behavior studied above applies to an environment in which there is a single decision maker with hyperbolic discounting (this continues to be a strategic environment in which the players are the different selves of the agent). In the working paper (Battaglini (2023)), I also illustrate how the analysis can be applied to dynamic multi-agent problems with free riding, and to cases in which policies are decided in a process of noncooperative bargaining as in Battaglini and Coate (2007, 2008) and Battaglini (2011).

In the case of a single decision maker with β-δ preferences as in Phelps and Pollak (1968) and Laibson (1997), the policy solves:[35]

$$\max_{y \geq (1-\gamma)x - l} \big\{K\big[y - (1-\gamma)x\big] - e(x) + \beta\delta v(y)\big\}, \tag{20}$$

where the only difference with (4) is that there is an additional term, the hyperbolic discount factor $\beta < 1$. The decision maker plays a game against his/her future selves. The expected continuation value $v(\cdot)$ must satisfy $v(x) = K(y(x) - (1-\gamma)x) - e(x) + \delta v(y(x)))$, where $y(x)$ is the expected future policy and β does not appear. This expression can be written as:

$$v(x) = \max_{y \geq (1-\gamma)x - l} \big\{K\big[y - (1-\gamma)x\big] - e(x) + \beta\delta v(y)\big\} + (1-\beta)\delta v\big(y(x)\big). \tag{21}$$

Condition (20) and (21) correspond to conditions (4) and (7) presented above. The second term in (21), $(1-\beta)\delta v(y(x))$, is the time inconsistency gap, i.e., the difference between the decision-maker's objective function and the expected value function. Because of this additional term, the shape of the expected value function directly depends on the expected future investment function $y(\cdot)$ as in (7). Differentiating (21) and using the first-order necessary condition $K/(\beta\delta) = -v'(x)$ from (20), we obtain a condition analogous to (11) in the analysis of Section 3:

$$y(x, c) = \left[\frac{1 - (1-\gamma)\delta\beta}{\delta(1-\beta)}\right] \cdot x - \frac{\beta}{(1-\beta)k} \cdot e(x) + c.$$

[35] The parameter β was used in Sections 3 and 4 in the definition of $e(x)$. We use it for the time preferences since this is the traditional notation for hyperbolic discounting and there is no risk of confusion since $e(\cdot)$ can be assumed here to be a general convex function.

5. General preferences

In the previous analysis, we made two simplifying assumptions on the functional form of the utility function. In (1), we assumed a quadratic cost function $e(\cdot)$ and, more importantly, quasilinear preferences. We discuss and relax these assumptions in this section.

5.1 *A sufficient condition for chaos with a general* $e(\cdot)$

The assumption of a quadratic $e(\cdot)$ allows us to provide a tractable solution for $\psi(\cdot, c)$ in (11) and an exhaustive characterization of the possible equilibrium dynamics. Dispensing this assumption, however, does not qualitatively change the analysis. As shown in (10), the differential equation (9) does not require a quadratic functional form to be solved and, for a generic convex and differentiable $e(\cdot)$, it generates a nonmonotonic, hump shaped investment function $\Psi(x, c)$, a feature that is key to obtain cycles and complex dynamics (see Figure 2 for an illustration). Whether such an investment function generates an attracting cycle or a chaotic trajectory depends on the shape of $e(\cdot)$.[36] To generalize the analysis, we only need a mild strengthening of the assumptions on $e(\cdot)$.

Assumption 1. *The function $\Psi(x, c)$ defined in (10) is strictly concave and thrice continuously differentiable with negative Schwarzian derivative with respect to x.*

Many common functions have negative Schwarzian derivative including, for instance, any polynomial of degree larger than or equal to 2 with real valued critical points (and thus the quadratic used in Section 4) and the exponential function.[37] Note that the specific value of c is irrelevant for Assumption 1, so it is just an assumption on the exogenous function $a_1 x - a_2 e(x)$, where $e(x)$ is exogenous and a_1 and a_2 are the coefficients in (10), defined by the parameters of the model. Let x^* be the critical point of $\Psi(\cdot, c)$ and define $\Delta^{2,3}(c) = [\Psi]^3(x^*, c) - [\Psi]^2(x^*, c)$. This is the gap between the third and the second iteration starting from the critical point x^* (i.e., the point at which the maximum is attained), which is an easily computed function of c given (10). We have the following.

Proposition 4. *Assume that $\Psi(x, c)$ satisfies Assumption 1 and there is a c such that $\Delta^{2,3}(c) < 0$, then there exists a c^* such that $\Psi(x, c^*)$ defined in (10) is topologically conjugate to $L_4(\cdot)$, and thus displays ergodic chaos on $[[\Psi]^2(x^*, c^*), \Psi(x^*, c^*)]$.*

Proof. See Section A.6 in the Appendix. □

Given Assumption 1, the requirement in Proposition 4 that there exists a c such that $\Delta^{2,3}(c) < 0$ is easy to satisfy, for example, it is always satisfied when $e(\cdot)$ is quadratic; and it is always satisfied if $[2/\delta - (1 + \alpha)(1 - \gamma)]/(1 - \alpha) > 1$ when $e(\cdot)$ is exponential. The following example illustrates Proposition 4 with an exponential cost function.

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[36] Li and Yorke (1975) have shown that, if continuous in x, a dynamical system $y(x)$ has cycles of any order in a set X if there is a $x' \in X$ such that $[y]^3(x') < x' < y(x') < [y]^2(x')$. Li and Yorke's condition is relatively easy to verify, but it does not generally guarantee that the region in which $\Psi(x, c)$ is chaotic (the scrambling set) has positive measure, leaving open the possibility that the chaotic region is reached only from a measure zero set of states.

[37] For a definition of the Schwarzian derivative, see Collet and Eckmann (1980, Section II.4).

EXAMPLE 1. Assume that $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$, $K = 2$, but $e(x) = 5\exp(x - 10)$. Using (10), it is easy to verify that for $c = 5.8$, $\Delta^{2,3}(c) = -300.99 < 0$. The value c^* is found solving $\Delta^{2,3}(c^*) = 0$, which gives us $c^* = 15.692453$. We thus have $\Psi(x, c) = 6.0263(x - 10) - 5\exp(x - 10) + 15.692453$. It is easy to verify that this function has a critical point at $x^* = 10.18670$, it is S-unimodal and chaotic in $[9.4301, 10.7912]$. ♢

The requirement in Proposition 4 that there exists a c such that $\Delta^{2,3}(c) < 0$ is necessary for the existence of a c such that $\Psi(\cdot, c)$ is topologically conjugate to $L_4(\cdot)$, and it is not always satisfiable. For example, when $e(\cdot)$ is exponential, if the parameters are such that $[2/\delta - (1 + \alpha)(1 - \gamma)]/(1 - \alpha) = 1/2$ and $2/[K(1 - \alpha)] = 1$, then $\Psi(\cdot, c)$ satisfies Assumption 1, but we have $\Delta^{2,3}(c) > 0$ for all c: indeed, in this case, $\Psi(\cdot, c)$ has a unique fixed point for all c, so it is never topologically conjugate to $L_4(\cdot)$.

Proposition 4 is a simple application of the equilibrium construction in Proposition 1 leading to (10), and of a theorem by Misiurewicz (1981). Misiurewicz's theorem proves that if a dynamical system y is S-unimodal and such that the iterates $[y]^n(x^*)$ of the critical point x^* converge to an unstable cycle, then y has exactly one absolutely continuous invariant measure.[38] Proposition 4 shows a condition under which at least a solution of (10) satisfies this property.

Clearly, the condition requiring $[y]^n(x^*)$ to enter an unstable cycle as $n \to \infty$ is a knife-edge condition. Proposition 4 however allows to construct equilibria with ergodic chaos that exist for *generic* parametrizations because c is endogenous: it can generically be selected to make sure that there is an n and a c^* such that $[\Psi]^n(x^*, c^*)$ enters an unstable cycle (in the construction of Proposition 4, an unstable steady state corresponding to the lowest fixed point of $\Psi(x, c)$). This is the key contribution of our paper. Note that the condition in Proposition 4 is only sufficient and indeed it can be easily extended using the same logic if we are willing to check conditions on higher iterates of $\Psi(x, c)$.[39]

5.2 *Other preferences*

Relaxing the assumption of quasilinear preferences has a more interesting impact on the analysis because, although the logic remains the same, it requires a generalization of the equilibrium construction. We discuss this point in the reminder of this section. Consider the more general utility functions described in Section 2 and define for convenience: $u(y, x) = u^{i,i}(y - (1 - \gamma)x, x)$, and $\phi(y, x) = u^{i,j}(y - (1 - \gamma)x, x)$ for i and $j \neq i$ to be respectively the citizens' utilities when their party is in office, and when their party is out of office. To study this environment it is useful to move beyond the special case of Markov equilibria and define a slightly more general strategy function $y(z, x)$ with one period memory, depending on the current state z and the precedent state x. The strategy defined in the previous section can be seen as a special case of $y(z, x)$.

[38] See Corollary 6 in Grandmont (1992) and Theorem II.8.3 and Corollary II.8.4 in Collet and Eckmann (1980).

[39] The condition in Proposition 4 is indeed sufficient to have the second iterate enter the repelling steady state $x^*_-(c)$. It is easy to verify that the second example in Figure 1 (the one constructed with Ruelle's constant) has the property that it is the third iterate to enters the repelling steady state $x^*_+(c)$.

The problem of the incumbent at $t-1$ in state $x_{t-1}=x$ is to find a state z that solves

$$\max_{z\geq(1-\gamma)x-l}\{u(z,x)+\delta v(z,x)\}. \tag{22}$$

where $v(z,x)$ is the expected value function evaluated at t when the state chosen at t is $x_t=z$, and the state at $t-1$ was $x_{t-1}=x$ (the state x_{t-2} is irrelevant for the set of solutions of (22), so it can be ignored for the discussion here). As in Section 3.1, $v(z,x)$ can be written as

$$\begin{aligned} v(z,x) &= \frac{1}{2}u\big(y(z,x),z\big)+\frac{1}{2}\phi\big(y(z,x),z\big)+\delta v\big(y(z,x),z\big) \\ &= u\big(y(z,x),z\big)+\delta v\big(y(z,x),z\big)-\Phi\big(y(z,x),z\big) \end{aligned} \tag{23}$$

where we define $\Phi(y(z,x),z)=(1/2)[u(y(z,x),z)-\phi(y(z,x),z)]$. Assuming (without loss of generality) differentiability, the envelope theorem allows us to differentiate the value function with respect to the first argument, z:

$$v_1(z,x)=u_2\big(y(z,x),z\big)-\Phi_1\big(y(z,x),z\big)y_1(z,x)-\Phi_2\big(y(z,x),z\big)+\delta v_2\big(y(z,x),z\big) \tag{24}$$

where v_q, u_q, and Φ_q for $q=1,2$ are the derivatives with respect to the qth arguments of v, u, and Φ. This condition still depends on the value function, through its derivative with respect to the second term. Using (23) again, we can see that this derivative in a state s, z can be written as

$$\begin{aligned} v_2(s,z) &= \big[u_1\big(y(s,z),s\big)+\delta v_1\big(y(s,z),s\big)\big]y_2(s,z)-\Phi_1\big(y(s,z),s\big)\cdot y_2(s,z) \\ &= -\Phi_1\big(y(s,z),s\big)\cdot y_2(s,z) \end{aligned}$$

where in the last equality we again apply the envelope theorem. Combining the equation above with (24), and using the first-order necessary condition from (22), we obtain

$$\begin{aligned} -\frac{u_1(z,x)}{\delta} &= u_2\big(y(z,x),z\big)-\Phi_1\big(y(z,x),z\big)y_1(z,x)-\Phi_2\big(y(z,x),z\big) \\ &\quad -\delta\Phi_1\big(y\big(y(z,x),z\big),y(z,x)\big)\cdot y_2\big(y(z,x),z\big) \end{aligned} \tag{25}$$

Condition (25) and the function $y(z,x)$ that satisfies it play the same role as condition (9) and $\Psi(\cdot,c)$ studied in Section 3.1. The key difference is that while (9) defines a simple differential equation, now the functional equation (25) defines a significantly more complex partial differential equation (PDE). The reason for this is intuitive. As the policy maker in x selects z at $t-1$, a marginal change in z affects the future in two ways. First, it affects the policy $y(z,x)$ chosen at t, which is a function of z as in Section 3. The policy change at t affects the expected value function at t because the envelope theorem does not fully apply given that the problem is time inconsistent. But now we have a novel second effect. A marginal change of the policy z at $t-1$ also affects the way in which the policy maker at $t+1$ reacts to the policy maker at t, i.e., for any choice $y(z,x)$ made at t, now a marginal change in z induces a policy change at $t+1$ by $y_2(y(z,x),z)$. Once again, part of this effect is "neutralized" by the envelope theorem applied at $t+1$, but part of it remains through the marginal effect on $\Phi(y(y(z,x),z),y(z,x))$.

The second effect described above is present because we have assumed a strategy $y(z, x)$ with one period memory. Condition (25) makes it clear why it is necessary to consider a strategy $y(z, x)$ with one period memory. When $u(z, x)$ is quasilinear as in the previous sections, $u_1(z, x)$ is independent of x. It follows that the right-hand side of (25) is also independent of x, implying that the equilibrium strategy satisfying (25) must be only a function of z. In the general case, however, $u_1(z, x)$ is a function of x, implying that we need to allow $y(z, x)$ to be a function of x as well to satisfy (25). The reason is that $y(z, x)$ is designed to make the policy maker in state x indifferent with respect to z: if the marginal utility of a change in z at $t-1$ depends on x, then future policy makers must adjust accordingly.

The functional equation (25) combines elements of a PDE, since it is a function of the partial derivatives $y_1(z, x)$ and $y_2(y(z, x), z)$; and elements of a difference equation, since the partial derivatives are evaluated at states $x_{t-1} = x$, $x_t = z$, and states $x_t = z$, $x_{t+1} = y(z, x)$. Because of this, (25) does not allow to have a simple analytical characterization, except for the quasilinear case studied in the previous section (which can be seen as just a special case of ((25))). The analysis of the previous section helps because it gives us a closed-form solution in the limit case with quasilinear utilities that can be used to compute numerical solutions. A numerical study of (25) with examples of cycles and aperiodic behavior is presented in the working paper (Battaglini (2023)).

6. Conclusions

We have studied a simple dynamic game in which in every period a politically motivated decision maker selects a policy that affects a state variable strategically linking policy-making periods. Because of political turnover, the preference of the policy maker may change over time, causing the decision process to be time inconsistent. We ask the question: under what conditions can such a simple model generate cycles and complex, unpredictable dynamics? Complex dynamics are impossible when policies are selected by a benevolent, time-consistent policy maker. In the presence of time inconsistency generated by the political process, however, simple sufficient conditions guarantee the existence of equilibria with cycles of any order and even chaos for generic economies. The degree of instability and unpredictability depends on the degree of time inconsistency: as time inconsistency converges to zero, chaotic equilibria continue to exist, but the size of the region containing the chaotic or cycling trajectories vanishes.

A limitation of our results is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but instead of the specific class of equilibria that we have characterized. Still, they show simple yet realistic environments in which predicting dynamic public policies is impossible in the sense that there are always chaotic equilibria that make it impossible. The problem is not that there are multiple equilibria, but that once we know the equilibrium, the dynamics are effectively unpredictable, even without random shocks to the system. Equilibria with complex dynamics, moreover, highlight a new source of inefficiency generated in political equilibria that has no correspondent in standard planner's problems: the instability of policies even in the absence of external shock.

Appendix

A.1 *Proof of Lemma 1*

For any real number c satisfying $c \geq (3-\varphi_1)(1+\varphi_1)/(4\varphi_2)$, define, as in Section 3, $x_-^*(c)$, $x_+^*(c)$ to be, respectively, the lowest and the largest fixed points of $\psi(x, c)$. We have

$$x_-^*(c) = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c}}{2\varphi_2}, \qquad x_+^*(c) = \frac{\varphi_1 - 1 + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c}}{2\varphi_2}.$$

We proceed in four steps.

Step 1. We first prove that for $c \in \mathcal{C}^*$, then $[\psi]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$, where $[\psi]^k(x, c)$ is the kth iteration of ψ, $[\psi]^k(x, c) = \psi(\psi^{k-1}(x, c), c)$. A sufficient condition for $[\psi]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$ is

$$\frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2\big(c - \underline{x}(c)\big)}}{2\varphi_2} \leq \frac{\varphi_1^2}{4\varphi_2} + c \leq \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2\big(c - \underline{x}(c)\big)}}{2\varphi_2}. \tag{26}$$

Note that $c \geq (3-\varphi_1)(1+\varphi_1)/(4\varphi_2)$ implies $\varphi_1^2/(4\varphi_2) + k \geq \varphi_1/(2\varphi_2)$. It follows that the first inequality in (26) is always satisfied. So, we need

$$\frac{\varphi_1^2}{4\varphi_2} + c - \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2\big(c - \underline{x}(c)\big)}}{2\varphi_2} \leq 0. \tag{27}$$

We now show that this condition is satisfied for any $c \in \mathcal{C}^*$. To this goal, we proceed by induction.

Step 1.1. Given $c \geq (3-\varphi_1)(1+\varphi_1)/(4\varphi_2) = c_0$, we have

$$\underline{x}(c) = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c}}{2\varphi_2} \leq \left(\frac{\varphi_1 - 3}{2\varphi_2}\right).$$

So, $[\psi]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$, if

$$\frac{\varphi_1^2}{4\varphi_2} + c \leq \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2\left(c - \dfrac{\varphi_1 - 3}{2\varphi_2}\right)}}{2\varphi_2}. \tag{28}$$

After a change in variable, (28) can be written as $\xi^2 - 2\xi - 6 \leq 0$, where

$$\xi = \sqrt{\varphi_1^2 + 4\varphi_2\big(c - (\varphi_1 - 3)/2\varphi_2\big)}.$$

It follows that we need $\sqrt{\varphi_1^2 + 4\varphi_2(c - (\varphi_1 - 3)/2\varphi_2)} \leq 1 + \sqrt{7}$, or $c \leq [3 + 2\sqrt{7} - (\varphi_1 - 1)^2]/(4\varphi_2)$. We therefore conclude that $[\psi]^2(\frac{\varphi_1}{2\varphi_2}, c) \geq \underline{x}(c)$ is satisfied for any

$$c \in \left[\frac{(3-\varphi_1)(1+\varphi_1)}{4\varphi_2}, \frac{3 + 2\sqrt{7} - (\varphi_1 - 1)^2}{4\varphi_2}\right],$$

which gives us a nonempty set.

Step 1.2. We now prove that if $[\psi]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$ in for any $c \in [(3-\varphi_1)(1+\varphi_1)/(4\varphi_2), c_n]$ for some $c_n < (4-\varphi_1)(2+\varphi_1)/(4\varphi_2)$, then we can find a $c_{n+1} > c_n$ such that the property $[\psi]^2(\varphi_1/2\varphi_2, c) \geq \underline{x}(c)$ is satisfied in $c \in [(3-\varphi_1)(1+\varphi_1)/(4\varphi_2), c_{n+1}]$. From the previous step, we know that this property is true for $c_1 = (3+2\sqrt{7}-(\varphi_1-1)^2)/4\varphi_2$. Let us assume we have proven it up to some $c_n \in [(3+2\sqrt{7}-(\varphi_1-1)^2)/(4\varphi_2), (4-\varphi_1)(2+\varphi_1)/(4\varphi_2))$. Note that if $c \geq c_n$, then we have

$$\underline{x}(c) \leq \frac{\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 c_n}}{2\varphi_2} = \frac{\varphi_1 - 2 - S_n}{2\varphi_2},$$

where $S_n = 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 c_n}$ is the slope of $\psi(x, c_n)$ at its largest fixed-point x_+^n.

Note that (27) is implied by

$$\varphi_1^2 + 4\varphi_2\left(c - \frac{\varphi_1 - 2 + S_n}{2\varphi_2}\right) - 2(2 - S_n) - 2\sqrt{\varphi_1^2 + 4\varphi_2\left(c - \frac{\varphi_1 - 2 + S_n}{2\varphi_2}\right)} \leq 0.$$

After a change in variable, this condition can be written as $\xi^2 - 2\xi - 2(2-S_n) \leq 0$, so $\xi \leq 1 + \sqrt{1 + 2(2-S_n)}$, where $\xi = \sqrt{\varphi_1^2 + 4\varphi_2(c - (\varphi_1 - 2 + S_n)/(2\varphi_2))}$. So, we need

$$\sqrt{\varphi_1^2 + 4\varphi_2\left(c - \frac{\varphi_1 - 2 + S_n}{2\varphi_2}\right)} \leq 1 + \sqrt{1 + 2(2-S_n)}$$

$$\Leftrightarrow \quad c \leq \frac{3 + 2\sqrt{3 + 2\sqrt{(\varphi_1-1)^2 + 4\varphi_2 c_n}} - (\varphi_1-1)^2}{4\varphi_2} = c_{n+1}.$$

We have the result if $c_{n+1} > c_n$. For this, we need $3 + 2\sqrt{3 + 2\sqrt{(\varphi_1-1)^2 + 4\varphi_2 c_n}} - (\varphi_1-1)^2 > 4\varphi_2 c_n$. It is easy to see that this inequality is satisfied for $4\varphi_2 c_n + (\varphi_1-1)^2 \leq 9$, or $c_n < (4-\varphi_1)(2+\varphi_1)/(4\varphi_2)$, which is always satisfied since we are assuming it in the induction step.

Step 1.3. The sequence c_n is bounded above by $(4-\varphi_1)(2+\varphi_1)/(4\varphi_2)$, thus it converges to $c_\infty = (4-\varphi_1)(2+\varphi_1)/(4\varphi_2)$. Since $[\psi]^2(\varphi_1/(2\varphi_2), c)$ is continuous in c, we have that $[\psi]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$ for any $c \in [(3-\varphi_1)(1+\varphi_1)/(4\varphi_2), (4-\varphi_1)(2+\varphi_1)/(4\varphi_2)]$, thus proving the result.

Step 2. We now prove that $\psi(x, c) \in X^*(c)$ for any x in $X^*(c)$ and c satisfying $c \in \mathcal{C}^*$. To see this, first note that for any $x \in X^*(c)$, we have $\psi(x, c) \leq \max_z \psi(z, c) = \psi(\varphi_1/(2\varphi_2), c) = \overline{x}(c)$, where the equality follows from the fact that $\psi(z, c)$ achieves a maximum at $\varphi_1/(2\varphi_2)$, and the second equality from the definition of $\overline{x}(c)$. Then note that for any $x \in X^*(c)$,

$$\psi(x, c) \geq \min_{z \in \{\underline{x}(c), \overline{x}(c)\}} \psi(z, c) \geq \min\{\psi(\overline{x}(c), c), \underline{x}(c)\} \geq \underline{x}(c), \tag{29}$$

where the first inequality follows from the concavity of $\psi(\cdot, c)$ in $X^*(c)$, and the second from $\psi(\underline{x}(c), c) = \underline{x}(c)$, and the last inequality from $\psi(\overline{x}(c), c) = [y]^2(\varphi_1/(2\varphi_2), c) \geq \underline{x}(c)$

when c satisfies $c \in \mathcal{C}^*$. We conclude that $\psi(x, c) \in X^*(c)$ for any x in $X^*(c)$. It can also be verified that for any initial condition x_0, the state eventually enters in $X^*(c)$ in at most a finite number of steps.

Step 3. We finally show that for any $c \in \mathcal{C}^*$, $\psi(\cdot, c)$ does not admit a stable steady state. Note that $\psi(x, c) = x$ at the points $x^*_-(c)$ and $x^*_+(c)$. We have that for $c \in \mathcal{C}^*$: $[\psi]'(x^*_+(c), c) = 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c} < -1$ and $[\psi]'(x^*_-(c), c) = 1 + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c} > 1$: so, neither $x^*_-(c)$ nor $x^*_+(c)$ are attracting steady states.

A.2 *Proof of Lemma 2*

Assume $R \geq R^*(\alpha)$, where $R^*(\alpha)$ is defined in (14). We show here that then $y^*(x, c)$ is feasible for all x. Define $x_l^-(c)$, $x_l^+(c)$ the points at which $\psi(x, c)$ intersects $(1-\gamma)x - l$, that is,

$$x_l^-(c) = \frac{\varphi_1 - (1-\gamma) - \sqrt{\left(\varphi_1 - (1-\gamma)\right)^2 + 4\varphi_2(c+l)}}{2\varphi_2},$$

$$x_l^+(c) = \frac{\varphi_1 - (1-\gamma) + \sqrt{\left(\varphi_1 - (1-\gamma)\right)^2 + 4\varphi_2(c+l)}}{2\varphi_2}.$$

It is immediate to verify that $\psi(x, c) \geq (1-\gamma)x - l$ for $x \in [x_l^-(c), x_l^+(c)]$. To prove feasibility, we can therefore focus on states $x < x_l^-(c)$ and $x > x_l^+(c)$. Consider the case of states $x > x_l^-(c)$ first. We can write:

$$x_l^-(c) - \underline{x}(c) = \frac{1}{2\varphi_2}\left[\gamma + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2 c} - \sqrt{\left(\varphi_1 - (1-\gamma)\right)^2 + 4\varphi_2(c+l)}\right]. \tag{30}$$

For feasibility, we need to have $x_l^-(c) - \underline{x}(c) \leq 0$. Note that

$$(\varphi_1 - 1)^2 + 4\varphi_2 c \leq \left(\varphi_1 - (1-\gamma)\right)^2 + 4\varphi_2(c+l) \Leftrightarrow \gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l \geq 0. \tag{31}$$

Assume first that (31) is satisfied. In this case, the square parenthesis in (30) is increasing in c and $x_l^-(c) - \underline{x}(c)$ can be bounded above inserting the upper bound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^-(c) - \underline{x}(c) &\leq \frac{1}{2\varphi_2}\Big[\gamma + \sqrt{(\varphi_1 - 1)^2 + 9 - (\varphi_1 - 1)^2} \\ &\quad - \sqrt{\left(\varphi_1 - (1-\gamma)\right)^2 + 4\varphi_2 l + 9 - (\varphi_1 - 1)^2}\Big] \\ &= \frac{1}{2\varphi_2}\left[\gamma + 3 - \sqrt{9 + \gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l}\right]. \end{aligned}$$

So, we have $x_l^-(c) - \underline{x}(c) \leq 0$ if $\varphi_1 \geq 4 - 2\varphi_2 l/\gamma$. Consider now the case: $\gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l < 0$. Now $x_l^-(c) - \underline{x}(c)$ can be bounded above inserting the lower bound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^-(c) - \underline{x}(c) &\leq \frac{1}{2\varphi_2}\big[\gamma + \sqrt{(\varphi_1-1)^2 + 4 - (\varphi_1-1)^2} \\ &\quad - \sqrt{\big(\varphi_1 - (1-\gamma)\big)^2 + 4\varphi_2 l + 4 - (\varphi_1-1)^2}\big] \\ &= \frac{1}{2\varphi_2}\big[\gamma + 2 - \sqrt{4 + \gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l}\big]. \end{aligned}$$

So, we have $x_l^-(c) - \underline{x}(c) \leq 0$ if $\varphi_1 \geq 3 - 2\varphi_2 l/\gamma$. It follows that a sufficient condition is that $\varphi_1 \geq 4 - 2\varphi_2 l/\gamma = \varphi_{11}^*$.

Consider now states $x > x_l^+(c)$. We can write

$$x_l^+(c) - \overline{x}(c) = \frac{1}{4\varphi_2}\big[2\varphi_1 - 2(1-\gamma) + 2\sqrt{\big(\varphi_1 - (1-\gamma)\big)^2 + 4\varphi_2(c+l)} - \big(\varphi_1^2 + 4\varphi_2 c\big)\big].$$

For feasibility, we need $x_l^+(c)(\varphi_1) - \overline{x}(c) \geq 0$. The right-hand side is concave in c, so it is minimized at one of the extremes. If the minimum is at the lower bound, we have

$$\begin{aligned} x_l^+(c) - \overline{x}(c) &= \frac{1}{4\varphi_2}\big[2\varphi_1 - 2(1-\gamma) + 2\sqrt{\big(\varphi_1 - (1-\gamma)\big)^2 + 4\varphi_2(c+l)} - \big(\varphi_1^2 + 4\varphi_2 c\big)\big] \\ &\geq \frac{1}{4\varphi_2}\big[-5 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1-1)\gamma + 4 + 4\varphi_2 l}\big]. \end{aligned}$$

It follows that $x_l^+(c) - \overline{x}(c) \geq 0$ if $\varphi_1 \geq \frac{3}{8\gamma}[3 - 4\gamma] - 2\varphi_2 l/\gamma$. If the minimum is at the upper bound, we have

$$x_l^+(c) - \overline{x}(c) \geq \frac{1}{4\varphi_2}\big[-10 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1-1)\gamma + 9 + 4\varphi_2 l}\big],$$

which can be written as $\varphi_1 \geq [8 - 4\gamma]/\gamma - 2\varphi_2 l/\gamma$. Therefore, a sufficient condition for $x_l^+(c) - \overline{x}(c) \geq 0$ is that $\varphi_1 \geq [8 - 4\gamma]/\gamma - 2\varphi_2 l/\gamma = \varphi_{12}^*$. Note that $\varphi_{12}^* - \varphi_{11}^* = 8(1/\gamma - 1) > 0$. We conclude that $y^*(x, c)$ is feasible if $\varphi_1 \geq \varphi_{12}^*$. The condition can be written as

$$\frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right] \geq \frac{1}{\gamma}[8 - 4\gamma] - 2\frac{l}{\gamma} \cdot \frac{\beta}{(1-\alpha)K} \tag{32}$$

Since $\gamma\widehat{x} \geq 0$ and $l > 0$, we have that $\gamma\widehat{x} + l \geq 0$ and we can rewrite (32) as

$$R = \frac{\beta}{K} \geq \frac{4\delta(1-\alpha)(2-\gamma) + \delta(1+\alpha)(1-\gamma)\gamma - 2\gamma}{2\delta(\widehat{x}\gamma + l)} = R^*(\alpha). \tag{33}$$

It follows that $\psi(x, c) \geq (1-\gamma)x - l$ for $x \in [\underline{x}(c), \overline{x}(c)]$ if $R \geq R^*(\alpha)$. Moreover, $y^*(x, c)$ obviously satisfies the constraint $y \geq (1-\gamma)x - l$ for $x > \overline{x}(c)$. Finally, we have that $(1-\gamma)x - l \leq (1-\gamma)\underline{x}(c) - l \leq \psi(\underline{x}(c), c) = y^*(x, c)$ in $x < \underline{x}(c)$.

To prove that, for any initial condition x_0, the state x_t enters $X^*(\alpha)$ in finite time, note that $y^*(x, c) = \max\{\psi(\overline{x}(c), c), (1-\gamma)x - l\}$ for $x \geq \overline{x}(c)$ where $\psi(\overline{x}(c), c) < \overline{x}(c)$. It follows that for any $x \geq \overline{x}(c)$ there is a $k \in [1, \infty)$ such that $[y^*]^k(x, c) \leq \overline{x}(c)$. Since $y^*(x, c) = \underline{x}(c)$ for $x \leq \underline{x}(c)$, it follows that for any $x \in \mathbb{R}$ there is a $k \in [1, \infty)$ such that

$[y^*]^k(x, c) \in X^*(c)$. To see that $y^*(x, c)$ admits no attracting steady state, note that the only possible candidates are $x^*_-(c)$ and $x^*_+(c)$. By Lemma 1, the the derivative at $x^*_+(c)$ is strictly lower than -1 and right derivative at $x^*_-(c)$ is strictly larger than 1.

A.3 *Proof of Lemma 3*

We proceed in three steps.

Step 1. We first observe that $y_c(\cdot) = \psi(\cdot, c)$, as defined in (11), is conjugate to $Q_k = x^2 + k$ for $k = (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 c$ by the homeomorphism $\xi : x \to \varphi_1/2 - \varphi_2 x$. To see this, note that $\xi \circ y_c(x) = \varphi_1/2 + \varphi_2^2 x^2 - \varphi_1\varphi_2 x - c\varphi_2$ and, moreover,

$$\begin{aligned} Q_k \circ \xi(x) &= [\varphi_1/2 - \varphi_2 x]^2 + (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 c \\ &= \varphi_1/2 + \varphi_2^2 x^2 - \varphi_2(\varphi_1 x + c) = \xi \circ y_c(x). \end{aligned}$$

So, we have $Q_k \circ \xi = \xi \circ y_c$. Similarly, we can show that L_η is conjugate to Q_k with $k = \eta/2(1 - \eta/2)$ by the homeomorphism $h_\eta = -\eta x + \eta/2$.

Step 2. Let us now define $c(\eta; \varphi_1, \varphi_2)$ such that $\eta/2(1 - \eta/2) = (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 \cdot c(\eta; \varphi_1, \varphi_2)$, that is,

$$c(\eta; \varphi_1, \varphi_2) = \frac{1}{\varphi_2}\big[(\varphi_1/2)(1 - \varphi_1/2) - \eta/2(1 - \eta/2)\big].$$

We can then write

$$\begin{aligned} L_\eta &= h_\eta^{-1} \circ Q_{\eta/2(1-\eta/2)} \circ h_\eta = h_\eta^{-1} \circ \big[\xi \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \xi^{-1}\big] \circ h_\eta \\ &= \big[h_\eta^{-1} \circ \xi\big] \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \big[\xi^{-1} \circ h_\eta\big] = \big[\xi^{-1} \circ h_\eta\big]^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \big[\xi^{-1} \circ h_\eta\big] \\ &= z_\eta^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \quad \Leftrightarrow \quad z_\eta \circ L_\eta = y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \end{aligned}$$

where $z_\eta = \xi^{-1} \circ h_\eta$. This implies that L_η is topologically conjugate to $\psi(x, c(\eta; \varphi_1, \varphi_2))$ through the homeomorphism z_η.

Step 3. From Proposition 1, $y^*(x, c(\eta; \varphi_1, \varphi_2))$ is a self-map in $X^*(c(\eta; \varphi_1, \varphi_2))$ and an equilibrium if $c(\eta; \varphi_1, \varphi_2) \in C^*$ as defined in (13). We have

$$\frac{1}{\varphi_2}\big[(\varphi_1/2)(1 - \varphi_1/2) - \eta/2(1 - \eta/2)\big] \geq \frac{4 - (\varphi_1 - 1)^2}{4\varphi_2} \quad \Leftrightarrow \quad \eta/2(1 - \eta/2) \leq -\frac{3}{4}.$$

Moreover, we need

$$\frac{1}{\varphi_2}\big[(\varphi_1/2)(1 - \varphi_1/2) - \eta/2(1 - \eta/2)\big] \leq \frac{9 - (\varphi_1 - 1)^2}{4\varphi_2} \quad \Leftrightarrow \quad \eta/2(1 - \eta/2) \geq -2.$$

We can therefore construct an equilibrium that is conjugate to L_η if $-2 \leq \eta/2(1 - \eta/2) \leq -3/4$. We conclude that for $3 \leq \eta \leq 4$, $y^*(x, c(\eta; \varphi_1, \varphi_2))$ in $X^*(c(\eta; \varphi_1, \varphi_2))$ is topologically conjugate to $L_\eta(\cdot)$ in $[0, L_\eta(1/2)]$.

A.4 *Proof of Proposition 2*

The result follows from the argument in the text. For the existence of values η_k in [3, 4] such that L_{η_k} has a stable cycle of period k or displays topological chaos, see Devaney (1989).

A.5 *Proof of Proposition 3*

The fact that we have a set of positive measures of values in $\mathcal{C}^*$ such that an equilibrium with ergodic distribution exists follows from Lemma 3 and the discussion in Section 3. We proceed to the characterization of the ergodic distribution for $c = c(4; \varphi_1, \varphi_2)$. Let μ be the measure that is invariant under L_4, so that $\mu = L_{4*}\mu$. It is well known that μ is the Arcsine law $\mu(x) = 1/(\pi\sqrt{x(1-x)})$ (see, e.g., Jakobson (1981)). Let us define the so called "push forward" measure $z_{4*}\mu$ by $z_{4*}\mu(X) := \mu(z_4^{-1}(X))$ for any set $X \subset \mathbb{R}$, where z_4 is the homeomorphism such that $y^* \circ z_4 = z_4 \circ L_4$, defined in the proof of Lemma 3. We have:

$$z_{4*}\mu = z_{4*}[L_{4*}\mu] = (z_4 \circ L_4)_*\mu = \left(y^* \circ z_4\right)_*\mu = \left(y^*\right)_*(z_{4*}\mu),$$

where in the second and fourth equalities we use the definition of the push forward measure, and in the third the fact that $y^* \circ z_4 = z_4 \circ L_4$. So. we have $(y^*)_*(z_{4*}\mu) = z_{4*}\mu$. To find $z_{4*}\mu$, note that $z_4 = (\varphi_1 - 4)/(2\varphi_2) + (4/\varphi_2)x$, so $x = \varphi_2 z_4/4 - (\varphi_1 - 4)/8$. Using the Perron–Frobenius operator, it follows that:

$$\mu^*(x, \alpha, R) = \frac{1}{\pi\sqrt{x(1-x)}\left|z_4'(x)\right|} = \frac{b(\alpha, R)}{4\pi \cdot \sqrt{\left(\frac{\varphi_2}{4}x - \frac{\varphi_1 - 4}{8}\right)\left(1 - \frac{\varphi_2}{4}x + \frac{\varphi_1 - 4}{8}\right)}}$$
$$= \frac{2R}{\pi(1-\alpha) \cdot \sqrt{16 - \left(\frac{2R}{(1-\alpha)}x - \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + 2R\widehat{x}\right]\right)^2}},$$

which gives us (19) in the statement of Proposition 3.

A.6 *Proof of Proposition 4*

Given Assumption 1, the function $\Psi(x, c)$ defined in (10) is strictly concave and thrice continuously differentiable with negative Schwarzian derivative in $[[\Psi]^2(x^*, c), \Psi(x^*, c)]$ for any c, where x^* is the critical point of $\Psi(x, c)$. Define $\widetilde{c}$ the point such that $\Psi(x^*; \widetilde{c}) = x^*$. If we select $c' > \widetilde{c}$ sufficiently close to $\widetilde{c}$, then $x_+^*(c)$, the fixed point on the right of x^*, is an attractive steady state and $[\Psi]^3(x^*, c') - [\Psi]^2(x^*, c') > 0$. By assumption, moreover, there is a c'' such that $[\Psi]^3(x^*, c'') - [\Psi]^2(x^*, c'') < 0$. It follows that, by continuity, there must be a $c^* \in [c', c'']$ such that $[\Psi]^2(x^*, c^*) = [\Psi]^3(x^*, c^*)$, and necessarily it must be that $\Psi(x^*, c^*) > x^*$. This implies that $\Psi(x, c^*)$ satisfies assumptions S1–S3, S4" and S5 of Corollary 6 in Grandmont (1992); and it is such that, starting from x^*, the state enters a repelling steady state at the second iteration. We conclude from Corollary 6 in Grandmont (1992) that $\Psi(x, c^*)$ has a unique absolutely continuous invariant ergodic measure in $[[\Psi]^2(x^*, c^*), \Psi(x^*, c^*)]$. Moreover, $\Psi(\cdot, c^*)$ has the same kneading sequence (the itinerary starting from $\Psi(x^*, c^*)$) as $L_4(x)$, so $\Psi(\cdot, c^*)$ on $[[\Psi]^2(x^*, c^*), \Psi(x^*, c^*)]$ is topologically conjugate to $L_4(\cdot)$ on $[0, 1]$ by Theorem II.6.1.A in Collet and Eckmann (1980).

References

Alesina, Alberto (1988), "Macroeconomics and politics." NBER Macroeconomic Annual, 13–62. [0245]

Alesina, Alberto and Guido Tabellini (1990), "A positive theory of fiscal deficits and government debt." *The Review of Economic Studies*, 57 (3), 403–414. [0242]

Bai, Jinhui H. and Roger Lagunoff (2011), "On the faustian dynamics of policy and political power." *Review of Economic Studies*, 78, 17–48. 2011. [0245]

[BBCD+] Banks, John, J. Brooks, G. Cairns, G. Davis, and P. Stacey (1992), "On Devaney's definition of chaos." *The American Mathematical Monthly*, 99 (4), 332–334. [0248]

Battaglini, Marco (2011), "Political economy of public debt." *Annual Review of Economics*, 3, 161–189. [0266]

Battaglini, Marco (2023), "Chaos and unpredictability with time inconsistent policy makers." Cornell University. Mimeo. [0247, 0266, 0270]

Battaglini, Marco and Stephen Coate (2007), "Inefficiency in legislative policy-making: A dynamic analysis." *American Economic Review*, 97 (1), 118–149. 2007. [0242, 0264, 0266]

Battaglini, Marco and Stephen Coate (2008), "A dynamic theory of public spending, taxation and debt." *American Economic Review*, 98 (1), 201–236. [0242, 0264, 0266]

Battaglini, Marco, Salvatore Nunnari, and Thomas Palfrey (2012), "Dynamic free riding with irreversible investments." National Bureau of Economic Research. Working paper No. 17926. [0245]

Battaglini, Marco, Salvatore Nunnari, and Thomas Palfrey (2014), "Dynamic free riding with irreversible investments." *American Economic Review*, 2014 (104), 2858–2871. 9. [0241, 0245]

Bechhoefer, John (1996), "The birth of period 3, revisited." *Mathematics Magazine*, 69 (2), 115–118. [0259]

Benedicks, Michael and Lennart Carleson (1985), "On iterations of $1 - ax^2$ on $(-1, 1)$." *Annals of Mathematics.*, 122 (1), 1–25. [0261]

Benhabib, Jess and Richard H. Day (1982), "A characterization of erratic dynamics in the overlapping generations model." *Journal of Economic Dynamics and Control*, 4, 37–55. [0244, 0262]

Bewley, Truman (1986), "Dynamic implications of the form of the budget constraint." In *Models of Economic Dynamics* (H. F. Sonnenschein, ed.). Springer-Verlag, Berlin, New York. [0244]

Bischi, Gian Italo, Lorenzo Cerboni Baiardi, Fabio Lamantia, and Davide Radi (2024), "Nonlinear dynamics and game-theoretic modeling in economics and finance." *Annals of Operation Research*. [0244]

Boldrin, Michele and Luigi Montrucchio (1986), "On the indeterminacy of capital accumulation paths." *Journal of Economic Theory*, 40, 26–39. [0244]

Boylan, Richard T., John Ledyard, and Richard McKelvey (1996), "Political competition in a model of economic growth: Some theoretical results." *Economic Theory*, 7, 191–205. [0245]

Cao, Dan and Ivan Werning (2018a), "Dynamic savings choices with disagreements." NBER wp. n. 22007. [0245]

Cao, Dan and Ivan Werning (2018b), "Savings and dissavings with hyperbolic discounting." *Econometrica*, 86 (3), 805–857. [0245]

Chatterjee, Satyajit and Burcu Eyigungor (2016), "Continuous Markov equilibria with quasi-geometric discounting." *Journal of Economic Theory*, 163, 467–494. [0245]

Collet, Pierre and Jean-Pierre Eckmann (1980), *Iterated Maps on the Interval as Dynamical Systems*. Birkhauser, Boston. [0263, 0267, 0268, 0277]

Day, Richard (1982), "Irregular growth cycles." *American Economic Review*, 72 (3), 406–414. [0244, 0262]

Day, Richard and Giulio Pianigiani (1991), "Statistical dynamics and control." *Journal of Economic Behavior and Organization*, 16, 37–83. [0244]

Day, Richard and Wayne Shafer (1987), "Ergodic fluctuations in deterministic economic models." *Journal of Economic Behavior and Organization*, 8, 339–361. [0244]

Deneckere, Raymond and Kenneth Judd (1992), "Cyclical and chaotic behavior in a dynamic equilibrium model." In *Cycles and Chaos in Economic Equilibrium* (J. Benhabib, ed.). Princeton University Press, Princeton. Chapter 14. [0244]

Devaney, Robert. L. (1989), *Chaotic Dynamical Systems*. Addison-Wesley. [0248, 0258, 0259, 0276]

Feller, William (1950), *An Introduction to Probability Theory and Its Applications, Volume I*. New York: Wiley. [0261]

Fershtman, Chaim and Shmuel Nitzan (1991), "Dynamic voluntary provision of public goods." *European Economic Review*, 35, 1057–1067. [0241]

[GRSS+] Gardini, Laura, Davide Radi, Noemi Schmitt, Iryna Sushko, and Frank Westerhoff (2022), "Perception of fundamental values and financial market dynamics: Mathematical insights from a 2D piecewise linear map." *SIAM Journal on Applied Dynamical Systems*, 21, 2314–2337. [0244]

Graczyk, Jacek and Grzegorz Światek (1997), "Generic hyperbolicity in the logistic family." *Annals of Mathematics*, 146 (1), 1–52. [0262]

Grandmont, Jean-Michel (1985), "On endogenous competitive business cycles." *Econometrica*, 53, 995–1045. [0244]

Grandmont, Jean-Michel (1992), "Periodic and aperiodic behavior in discrete one-dimensional dynamical systems." In *Cycles and Chaos in Economic Equilibrium* (J. Benhabib, ed.). Princeton University Press, Princeton. [0268, 0276]

Harris, Christopher and David Laibson (2001), "Dynamic choices of hyperbolic consumers." *Econometrica*, 69 (4), 935–957. [0245]

Hommes, Cars (2013), *Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems*. Cambridge University Press, Oxford, UK. [0244]

Jakobson, Magnus V. (1981), "Absolutely continuous invariant measures for one-parameter families of one-dimensional maps." *Communication in Mathematical Physics*, 81, 39–88. [0258, 0261, 0276]

Khan, Ali and Tapan Mitra (2005), "On topological chaos in the Robinson–Solow–Srinivasan model." *Economic Letters*, 88, 127–133. [0244]

Klein, Paul, Per Krusell, and Jose-Victor Rios-Rull (2008), "Time-consistent public policy." *Review of Economic Studies*, 75, 789–808. [0242]

Krusell, Per and Antony A. Jr. Smith (2003), "Consumption-savings decisions with quasi-geometric discounting." *Econometrica*, 71 (1), 53–73. [0245]

Laibson, David (1997), "Golden eggs and hyperbolic discounting." *Quarterly Journal of Economics*, 112 (2), 443–477. [0266]

Lasota, Andrzej and James A. Yorke (1973), "On the existence of invariant measures for piecewise monotonic trasformations." *Transactions of the American Mathematical Society*, 186, 481–488. [0244]

Levhari, David and Leonard J. Mirman (1980), "The great fish war: An example using Nash—Cournot solution." *Bell Journal of Economics*, 11, 322–334. [0241]

Li, Tien-Yien and James A. Yorke (1975), "Period three implies chaos." *American Mathematical Monthly*, 82, 985–992. [0244, 0267]

Majumdar, Mukul, Tapan Mitra, and Kazuo Nishimura (2000), *Optimization and Chaos*. Springer. [0244]

Marx, Leslie M. and Stephen Matthews (2000), "Dynamic voluntary contribution to a public project." *The Review of Economic Studies*, 67, 327–358. [0241]

Matsuyama, Kiminori (1999), "Growing thorugh cycles." *Econometrica*, 67 (2), 335–347. [0244]

Matsuyama, Kiminori, Iryna Sushko, and Laura Gardini (2016), "Revisiting the model of credit cycles with good and bad projects." *Journal of Economic Theory*, 162, 225–256. [0244]

Melese, Francois and William Transue (1986), "Unscrambling chaos through thick and thin." *The Quarterly Journal of Economics*, 101 (2), 419–424. [0244]

Mignot, Sarah, Fabio Tramontana, and Frank Westerhoff (2023), "Complex dynamics in a nonlinear duopoly model with heterogeneous expectation formation and learning behavior." *Annals of Operations Research*. [0244]

Misiurewicz, Michal (1981), "Absolutely continuous measures for certain maps of an interval." *Publications Mathématiques de l'IHES*, 53, 17–51. [0263, 0268]

Morris, Stephen and Andrew Postlewaite (1997), "Observational implications of non-exponential discounting." Mimeo. [0245]

Persson, Torsten and Lars Svensson (1989), "Why a stubborn conservative would run a deficit: Policy with time-inconsistent preferences." *Quarterly Journal of Economics*, 104, 325–345. [0242]

Phelps, Edmund S. and Robert A. Pollak (1968), "On second-best national saving and game equilibrium growth." *Review of Economic Studies*, 35, 185–199. [0244, 0266]

Rosser, J. Barkley Jr. (2021), *Foundations and Applications of Complexity in Economics*. Springer, Berlin/Heidelberg, Germany. [0244]

Ruelle, David (1977), "Applications conservant une mesure absolument continue par rapport a' dx sur [0, 1)." *Communications in Mathematical Physics*, 55, 47–51. [0258, 0259]

Ulam, Stanislaw and John von Neumann (1947), "On combination of stochastic and deterministic processes." *Bulletin of the American Mathematical Society*, 53, 1120. [0258, 0259, 0260]

Vellekoop, Michel and Raoul Berglund (1994), "On intervals, transitivity = chaos." *American Mathematical Monthly*, 101 (4), 353–355. [0248]

Woodford, Michael (1988), "Imperfect financial intermediator and complex dynamics." In *Chaos, Sunspots, Bubbles and Nonlinearity* (W. Barnett, J. Geweke, and K. Shell, eds.). Cambridge University Press, New York. [0244]

Yared, Pierre (2010), "Politicians, taxes, and debt." *Review of Economic Studies*, 77, 806–840. [0242]

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