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On Groves mechanisms for costly inclusion

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We investigate Groves mechanisms for economies where (i) a social outcome specifies a group of winning agents, and (ii) a cost function associates each group with a monetary cost. In particular, we characterize both (i) the class of cost functions for which there are Groves mechanisms such that the agents cover the costs through voluntary payments, and (ii) the class of cost functions for which there are envy-free Groves mechanisms. It follows directly from our results that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments.

Keywords. Costly inclusion, Groves mechanism, pivot mechanism, Vickrey auction, free-rider problem, labor markets.

JEL classification. D24, D44, D47, D61, D63, D82, H41.

1. Introduction

We consider the problem of selecting a group of agents to receive a service and then allocating the associated costs, a remarkably prevalent problem faced by a variety of public

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enterprises and private firms. Across the rich collection of applications considered in the literature,[1] some common themes have emerged: the group of beneficiaries should maximize surplus (*production efficiency*), the costs should be covered without outside subsidy (*no-deficit*), nobody should contribute more than he is willing (*voluntary participation*), and the allocation should be fair (which can be formalized many ways; we consider *no-envy*). In this article, we thoroughly investigate the implementation of these objectives in dominant strategy equilibrium (*strategy-proofness*) when costs are known but valuations are private information.

For the first-best scenario—where both costs and valuations are known—it is a classical insight that in order to maximize some notion of social welfare, (i) marginal cost pricing is optimal when costs are convex and symmetric, due to standard arguments involving supply and demand; and (ii) deviations from marginal cost pricing may be optimal otherwise, which has been particularly investigated in the context of pricing public utilities (Ramsey (1927); Manne (1952); Boiteux (1956); Baumol and Bradford (1970)). Moreover, if it is known that each agent's valuation is sufficiently high, then the cost of serving all agents can be directly allocated on the basis of normative principles; this is the approach taken in the large literature on axiomatic cost allocation (see Young (1985a) and Algaba, Fragnelli, and Sánchez-Soriano (2019) for overviews). That said, these approaches are problematic when valuations are private information, as we cannot directly set prices using demand information and it may be inefficient to serve all agents unconditionally.

For the scenario we consider—where costs are known but valuations are not—there is an active literature on mechanism design for costly inclusion environments that has grown out of its seminal contributions (Young (1985b); Young (1998); Moulin (1999); Moulin and Shenker (2001)). It is well known that the mechanisms satisfying *production efficiency* and *strategy-proofness* are precisely the Groves mechanisms in a variety of models, including this one (Groves (1973); Green and Laffont (1979); Holmström (1979)), and early in this literature it was established that when the cost function is both monotone and concave (but not additive), no Groves mechanism satisfies *no-deficit*, satisfies *voluntary participation*, and never assigns positive transfers (Moulin and Shenker (2001)). Since then, the literature has primarily focused on monotone and concave cost functions (possibly with additional structure) while abandoning *production efficiency* in favor of precisely equating total contributions to total costs (Deb and Razzolini (1999a); Deb and Razzolini (1999b); Ohseto (2000); Mutuswami (2004); Ohseto (2005); Yu (2007); Ohseto (2009); Massó, Nicolò, Sen, Sharma, and Ülkü (2015); Hashimoto and Saitoh (2016)).[2]

[1]Examples include the overhead costs faced by branches of a firm (Shubik (1962)), the runway costs faced by airlines (Littlechild and Owen (1973)), the costs of common water resources such as multipurpose reservoirs and Tennessee Valley Authority projects (Straffin and Heaney (1981); Young, Okada, and Hashimoto (1982)), and the costs of networks such as those for cable television (Sharkey (1995)) and multicast messages (Herzog, Shenker, and Estrin (1997); Feigenbaum, Papadimitriou, and Shenker (2001)).

[2]As notable exceptions, *group strategy-proofness* has been investigated without any cost function restrictions (Juarez (2013)), and approximate versions of axioms have been thoroughly investigated by computer scientists (see Feigenbaum and Shenker (2004) for an overview).

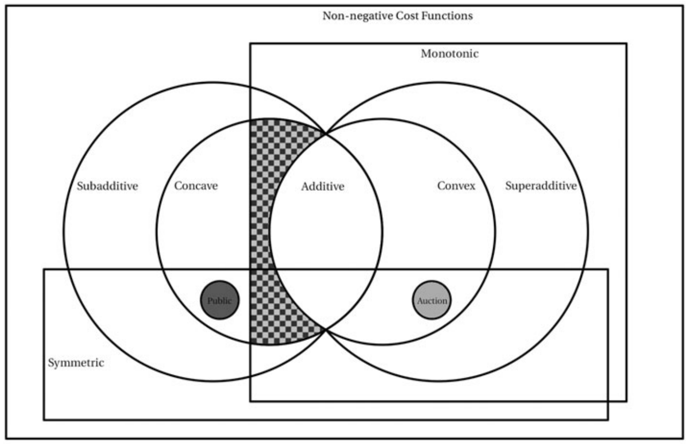
A few simple observations suggest that Groves mechanisms merit further investigation for costly inclusion environments. First, if costs are additive, then the classical prescription of marginal cost pricing (adapted to allow for personalized prices) trivially yields a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and that moreover satisfies *no-envy* when costs are symmetric: simply ask each agent if he is willing to pay his constant inclusion cost. Second, if we expand the model to allow for infinite costs modeling infeasibility, then we include auctions with identical objects where agents have unit demand, in which case the Vickrey mechanism (Vickrey (1961)) is a Groves mechanism that achieves all of our objectives.[3] These positive observations are mitigated by the fact that the expanded model also allows for binary public goods, for which a well-known version of the free-rider problem states that no Groves mechanism satisfies *no-deficit* and *voluntary participation* (see, for example, Green and Laffont (1979)). Taken together, these observations establish that Groves mechanisms have a limited but nontrivial scope for success. What, then, is this scope?

To answer this question, we introduce *inclusion cost coverage* as a generalization of nondecreasing average costs from quantity-based cost functions to set-based cost functions; a similar generalization was done in the context of cooperative games for convexity (Shapley (1971)). We also introduce *exclusion pivot mechanisms*, variants of standard pivot mechanisms[4] that differ only in that when an agent's peers maximize their surplus without him, they are restricted to selecting a group that excludes him. First, we show that the cost function satisfies inclusion cost coverage if and only if there are Groves mechanisms that satisfy *no-deficit* and *voluntary participation* if and only if each exclusion pivot mechanism is such a mechanism (Theorem 1). Second, we show that the cost function is convex and symmetric if and only if there are *envy-free* Groves mechanisms if and only if each exclusion pivot mechanism is such a mechanism (Theorem 2). Because convexity is a stronger requirement than inclusion cost coverage, it follows directly that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments (see Figure 1). Remarkably, this is precisely the case of monotonic supply curves first considered by the classical literature, and our proof involves establishing that for these environments, the exclusion pivot mechanisms have considerable structure involving the supply curve and the *reported* demand curve.

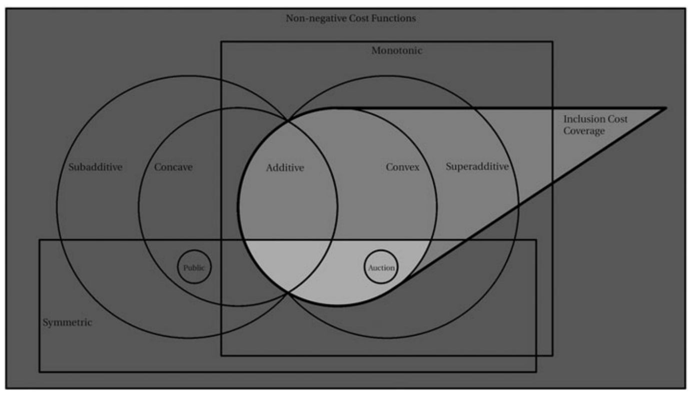
Altogether, our results both (i) reinforce the justification for abandoning *production efficiency* in favor of balancing the budget when costs are concave, and (ii) suggest Groves mechanisms merit further study when costs are convex. Our findings apply to

[3]In fact, there is a large literature on fair Groves mechanisms for environments with indivisible objects and money. Our paper is most closely related to those that have investigated *envy-free* Groves mechanisms (Pápai (2003); Ohseto (2006); Yengin (2012); Yengin (2017)); the primary difference is that we allow for production.

[4]The family of exclusion pivot mechanisms is essentially single-valued: each exclusion pivot mechanism is distinguished by a method of breaking ties when multiple groups maximize surplus, which has no welfare consequences. Similarly, the family of standard pivot mechanisms is essentially single-valued. We remark that in other models, it is common to refer to the standard pivot mechanism as the VCG mechanism (Vickrey (1961); Clarke (1971); Groves (1973)).



(a) Previous results from the literature.



(b) New results.

FIGURE 1. *Summary of results.* (a) For simple auction environments, the Vickrey mechanism is a Groves mechanism that satisfies *no-deficit*, *voluntary participation*, and *no-envy* (Vickrey (1961)). When providing a binary public good, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*; this is a version of the classic free-rider problem (see, for example, Green and Laffont (1979)). When costs are monotone and concave (but not additive), no Groves mechanism satisfies *no-deficit*, satisfies *voluntary participation*, and never assigns positive transfers (Moulin and Shenker (2001)). (b) For any cost function in the dark gray region, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. For any cost function in the medium gray region, the exclusion pivot mechanisms satisfy *no-deficit* and *voluntary participation*, but no Groves mechanism satisfies *no-envy*. For any cost function in the light gray region, the exclusion pivot mechanisms satisfy *no-deficit*, *voluntary participation*, and *no-envy*. We establish logical relations justifying how the medium gray region is drawn in Appendix C.

auctions, public goods, natural monopolies, two-sided markets, minimum cost spanning trees, and labor markets that involve revenue functions instead of cost functions; see Section 4.

2. Model

2.1 *Costly inclusion environments*

We consider the following class of production economies with incomplete information and no consumption externalities.

Definition. A *(costly inclusion) environment* is a pair (N, C), where

- N is a finite and nonempty set of *agents*. For convenience, we write $N = \{1, 2, \dots, n\}$.
- For each $i \in N$, a *bundle* $(t_i, x_i) \in \mathbb{R} \times \{0, 1\}$ specifies (i) a transfer of *money*, and (ii) whether or not i is a *winner*, with 1 representing that he is. An *allocation* $(t, W) \in \mathbb{R}^N \times 2^N$ specifies both a list of monetary transfers t and a group of winners W; this can also be written $(t, x) \in \mathbb{R}^N \times \{0, 1\}^N$ to emphasize the bundle each agent receives.
- Money can be used to produce a group of *winners*; we call an agent who does not win a *loser*. This is summarized by the *cost function*, $C : 2^N \to \mathbb{R} \cup \{\infty\}$ with $C(\emptyset) = 0$, which specifies the monetary cost of producing each winning group. We let $\mathcal{W}$ denote the collection of groups with finite cost.[5]
- Each $i \in N$ has a *valuation* $v_i \in \mathbb{R}$, which represents the preference relation R_i over bundles such that for each pair $(t_i, x_i), (t'_i, x'_i) \in \mathbb{R} \times \{0, 1\}$,

  $$(t_i, x_i)\, R_i\, \left(t'_i, x'_i\right) \quad \text{if and only if} \quad x_i v_i + t_i \geq x'_i v_i + t'_i.$$

  Each agent's valuation is his private information. We use V_i to denote $\mathbb{R}$ when emphasizing that it is the set of possible valuations for i, and we use $V \equiv \times V_i$ to denote the set of possible valuation profiles.[6]

For convenience, whenever we refer to a generic environment we will implicitly assume all notation introduced thus far.

Costly inclusion environments use infinite costs to model infeasible groups and negative costs to model revenues, and therefore cover many familiar environments from the literature. For example, if there is $k \in \{1, 2, \dots, n\}$ such that for each $W \subseteq N$,

$$C(W) = \begin{cases} 0, & |W| \leq k, \\ \infty, & |W| > k, \end{cases}$$

then we have an auction environment with k identical objects where agents have unit demand. As a second example, if there is $\kappa \in [0, \infty)$ such that

$$C(W) = \begin{cases} 0, & W = \emptyset, \\ \kappa, & W = N, \\ \infty, & \text{else}, \end{cases}$$

[5] We use ∞ to denote positive infinity and $-\infty$ to denote negative infinity; thus for each $x \in \mathbb{R}$, we have $\infty > x > -\infty$. We are careful to only use addition and subtraction for real numbers.

[6] We consider environments where each agent's set of possible valuations is $\mathbb{R}_+$ in Section 4.3.

then we have a public good environment. Finally, if all costs are negative, and if the associated revenue function satisfies gross substitutes, then we have a Kelso–Crawford labor market environment with a single firm (Kelso and Crawford (1982)). We discuss the implications of our results for these examples and others in Section 4.

2.2 *Mechanisms and axioms*

Because we focus on dominant strategy implementation, it is without loss of generality to focus on *(direct) mechanisms*, where the agents simultaneously report their preferences to an administrator who processes the reports to determine an allocation (Gibbard (1973); Myerson (1981)).

Definition. Fix an environment. A *transfer policy* is a mapping $\tau : V \to \mathbb{R}^N$. A *winner policy* is a mapping $\varphi : V \to 2^N$. A *mechanism* (τ, φ) consists of a transfer policy τ and a winner policy φ. Given a winner policy φ and an agent i, we sometimes speak of the associated policy for determining whether or not i is a winner, using $\varphi_i : V \to \{0, 1\}$ for the mapping

$$\varphi_i(v) \equiv \begin{cases} 1, & i \in \varphi(v), \\ 0, & i \notin \varphi(v). \end{cases}$$

We are interested in mechanisms that satisfy desirable properties, or *axioms*. In particular, we consider the following five standard axioms.

Definition. Fix an environment. For each $v \in V$ and each $W \subseteq N$, the *surplus of* W *at* v, $\sigma_v(W)$, is given by

$$\sigma_v(W) \equiv \begin{cases} \sum_{i \in W} v_i - C(W), & W \in \mathcal{W}, \\ -\infty, & \text{else}. \end{cases}$$

A mechanism (τ, φ) satisfies

- *production efficiency* if and only if for each $v \in V$ and each $W \subseteq N$, we have $\sigma_v(\varphi(v)) \geq \sigma_v(W)$;
- *strategy-proofness* if and only if for each $i \in N$, each $v \in V$, and each $v_i' \in V_i$, we have $\varphi_i(v)v_i + \tau_i(v) \geq \varphi_i(v_i', v_{-i})v_i + \tau_i(v_i', v_{-i})$;
- *no-deficit* if and only if for each $v \in V$, we have $0 \geq \sum \tau_i(v) + C(\varphi(v))$;
- *voluntary participation* if and only if for each $i \in N$ and each $v \in V$, we have $\varphi_i(v)v_i + \tau_i(v) \geq 0$; and
- *no-envy* if and only if for each pair $i, j \in N$ and each $v \in V$, we have $\varphi_i(v)v_i + \tau_i(v) \geq \varphi_j(v)v_i + \tau_j(v)$.

We say that φ is *surplus-maximizing* if and only if (τ, φ) satisfies *production efficiency*.

It follows from Holmström (1979) that the class of mechanisms satisfying *production efficiency* and *strategy-proofness* is precisely the class of Groves mechanisms; see Appendix A for the familiar definition. We are interested in (i) identifying the costly inclusion environments that have Groves mechanisms satisfying *no-deficit* and *voluntary participation*, and (ii) identifying the costly inclusion environments that have *envy-free* Groves mechanisms.

3. Results

3.1 *Preliminaries*

The collections of axioms we investigate are logically compatible for some environments but not for others, and this compatibility is determined by certain properties of the cost function. We introduce the properties that prove focal to our analysis here.

Definition. Fix an environment. For each nonempty $W \subseteq N$ and each $i \in W$, define the *inclusion cost for i in W*, $\mathcal{I}_i(W)$, by

$$\mathcal{I}_i(W) \equiv \begin{cases} C(W) - C\big(W\backslash\{i\}\big), & W, W\backslash\{i\} \in \mathcal{W}, \\ \infty, & W \notin \mathcal{W}, \\ -\infty, & \text{else.} \end{cases}$$

The cost function satisfies

- *convexity* if and only if for each $i \in N$ and each pair $W, W' \subseteq N$ such that $\{i\} \subseteq W \subseteq W'$, we have $\mathcal{I}_i(W') \geq \mathcal{I}_i(W)$;
- *symmetry* if and only if for each pair $W, W' \subseteq N$ such that $|W| = |W'|$, we have $C(W) = C(W')$; and
- *inclusion cost coverage* if and only if for each nonempty $W \in \mathcal{W}$, we have
  (i) for each $i \in W$, $W\backslash\{i\} \in \mathcal{W}$, and
  (ii) $\sum_{i \in W} \mathcal{I}_i(W) \geq C(W)$.

Both *convexity* and *symmetry* are standard properties, while *inclusion cost coverage* is novel. It is easy to verify that *inclusion cost coverage* is (i) satisfied by any auction environment with identical objects where agents have unit demand, and (ii) violated by any public good environment. Moreover, as we discuss in Section 4.2, it follows from results in the literature that *inclusion cost coverage* is satisfied by any Kelso–Crawford labor market environment with a single firm.

In Appendix C, we investigate the relationship between these cost function properties and three additional standard properties: *monotonicity*, *superadditivity*, and (for *symmetric* cost functions) *nondecreasing average costs*. In particular, we establish seven propositions providing logical relationships among these cost function properties, most of which are illustrated in Figure 1.

Both of our main results also involve variants of the classic *pivot mechanism* (Clarke (1971)), where each agent pays the difference between (i) the maximum surplus his peers could receive were he absent, and (ii) the surplus his peers do in fact receive. To highlight the distinction, we define both the standard version and our variants.

Definition. Fix an environment. A mechanism (τ, φ) is a *standard pivot mechanism* if and only if (i) φ is surplus-maximizing; and (ii) for each $i \in N$ and each $v \in V$,

$$\tau_i(v) = \left(\left[\max_{W \subseteq N} \sigma_v(W)\right] - \varphi_i(v)v_i\right) - \left[\max_{W \subseteq N} \sigma_{(0,v_{-i})}(W)\right].$$

A mechanism (τ, φ) is an *exclusion pivot mechanism* if and only if (i) φ is surplus-maximizing; and (ii) for each $i \in N$ and each $v \in V$,

$$\tau_i(v) = \left(\left[\max_{W \subseteq N} \sigma_v(W)\right] - \varphi_i(v)v_i\right) - \left[\max_{W \subseteq N\setminus\{i\}} \sigma_{(0,v_{-i})}(W)\right].$$

The two kinds of pivot mechanisms differ in how they calculate the maximum surplus that the peers of i could receive were he absent. In particular, though in both cases the calculation treats the valuation of i as zero by using the profile $(0, v_{-i})$, the mechanisms differ in what they allow the peers of i to select: a standard pivot mechanism allows them to select any social alternative, while an exclusion pivot mechanism requires them to select a group that excludes i. We remark that due to this difference, standard pivot mechanisms are defined for more general environments where social alternatives are not groups of agents, while exclusion pivot mechanisms are not. It is not hard to see that standard pivot mechanisms coincide with exclusion pivot mechanisms when costs are monotonic, as in this case the peers of i cannot improve their own surplus by including i, but the two versions differ in general—for example, they differ for public good provision.

Finally, each of our main results concerns one subcollection of our five axioms, and we frequently refer to these subcollections in our proof sketches and proofs. For convenience, we introduce the following shorthand terms for these sections.

Definition. A mechanism is

- *autonomous* if and only if it satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*; and
- *equitable* if and only if it satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

Equivalently, an *autonomous* mechanism is a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and an *equitable* mechanism is an *envy-free* Groves mechanism.

3.2 *Characterizations*

Our first result characterizes the costly inclusion environments for which there are Groves mechanisms satisfying *no-deficit* and *voluntary participation*.

Theorem 1. *Fix an environment. The following are equivalent:*

- *there is a mechanism that satisfies* production efficiency, strategy-proofness, no-deficit, *and* voluntary participation*;*
- *each exclusion pivot mechanism satisfies* production efficiency, strategy-proofness, no-deficit, *and* voluntary participation*; and*
- *the cost function satisfies* inclusion cost coverage.

Proof sketch. See Appendix A for the formal proof and for more detailed proof sketches of the lemmas. Recall that for our proofs, we use *autonomous* mechanism as a shorthand for Groves mechanism satisfying *no-deficit* and *voluntary participation*.

The proof consists of four lemmas, and involves simplifying our problem to characterizing the cost functions for which exclusion pivot mechanisms satisfy *no-deficit*. In particular, we begin by verifying that the exclusion pivot mechanisms are always Groves mechanisms that satisfy *voluntary participation* (Lemma A.1), then prove that whenever any such mechanism assigns low enough transfers to satisfy *no-deficit*, necessarily each exclusion pivot mechanism does as well (Lemma A.2). To establish that *inclusion cost coverage* is sufficient for the existence of an *autonomous* mechanism, we show that for these cost functions each exclusion pivot mechanism always requires each loser to pay nothing and each winner to pay at least his inclusion cost, and therefore satisfies *no-deficit* and is *autonomous* (Lemma A.3). This leaves open the possibility that exclusion pivot mechanisms might be able to cover costs more often by charging winners *more* than their inclusion costs whenever necessary, but to complete the proof we argue that in fact if *inclusion cost coverage* is violated, then no exclusion pivot mechanism—and therefore no mechanism at all—is *autonomous* (Lemma A.4).

Before proceeding, we provide some intuition for the proof of Lemma A.4; see this point in the working paper version (Mackenzie and Trudeau (2022)) for a more detailed proof sketch of the entire theorem. To begin, observe that if all costs are finite, then at any profile where all winners have extremely high valuations and all losers have extremely low valuations, any exclusion pivot mechanism charges each winner precisely his inclusion cost and charges each loser nothing; it follows that if all costs are finite, then any cost function that violates *inclusion cost coverage* has no *autonomous* mechanism. That said, this argument does not work for a public good cost function, such as the following example for $N = \{1, 2, 3\}$:

$$C(W) \equiv \begin{cases} 0, & W = \emptyset, \\ \infty, & \emptyset \subsetneq W \subsetneq N, \\ 5, & W = N. \end{cases}$$

For this example, we cannot construct profiles where we select N and charge winners their inclusion costs because these inclusion costs are each negative infinity. Notice, however, that at the profile $(2, 2, 2)$, each agent pays 1—which is already not enough to cover costs—while at the profile $(100, 100, 100)$, matters are strictly worse as each agent *receives* 195. Indeed, as all valuations increase, each winner is more desperately needed by the others, and is thus compensated by the exclusion pivot mechanisms accordingly. In the formal proof, we generalize this insight to arbitrary cost functions to show that whenever *inclusion cost coverage* is violated, each exclusion pivot mechanism runs a deficit at some profile. □

Our second result characterizes the costly inclusion environments for which there are *envy-free* Groves mechanisms.

THEOREM 2. *Fix an environment. The following are equivalent:*

- *there is a mechanism that satisfies* production efficiency*,* strategy-proofness*, and* no-envy*;*
- *each exclusion pivot mechanism satisfies* production efficiency*,* strategy-proofness*, and* no-envy*; and*
- *the cost function satisfies* convexity *and* symmetry.

PROOF SKETCH. See Appendix B for the formal proof and for more detailed proof sketches of the lemmas. Recall that for our proofs, we use *equitable* mechanism as a shorthand for *envy-free* Groves mechanism.

The proof consists of three lemmas, and in contrast to the proof of our first theorem does *not* begin by simplifying our problem to the study of exclusion pivot mechanisms. Instead, we partition the set of cost functions into (i) those that violate *convexity*, (ii) those that satisfy *convexity* but violate *symmetry*, and (iii) those that satisfy both *convexity* and *symmetry*. In particular, we first argue that for any cost function in the first class, we can identify a situation where one agent controls whether or not another wins regardless of whether the latter changes his report slightly, and argue that no Groves mechanism can address such a situation in an *envy-free* manner (Lemma B.1). Second, we argue that for any cost function in the second class, we can identify a situation where surplus is maximized by including one agent and excluding another even though the latter has a higher valuation, which is incompatible with *no-envy* (Lemma B.2). Finally, we argue that for each cost function in the third class, each exclusion pivot mechanism has considerable structure involving the supply curve and the reported demand curve: if $\hat{q}$ is the maximum number of winners compatible with surplus maximization, then either (i) each winner pays the $\hat{q}$th-lowest marginal cost, or (ii) each winner pays the $(\hat{q}+1)$th-highest valuation, and in both cases there is no envy (Lemma B.3). These are three logically distinct arguments; none of these lemmas is proved using one of the others.

Before proceeding, we provide some intuition for the proof of Lemma B.1; see this point in the working paper version (Mackenzie and Trudeau (2022)) for a more detailed proof sketch of the entire theorem. Consider the following example for $N = \{1, 2, 3\}$, which satisfies *symmetry* and *inclusion cost coverage* but violates *convexity*:

$$C(W) \equiv \begin{cases} 0, & |W| = 0, \\ 1, & |W| = 1, \\ 8, & |W| = 2, \\ 12, & |W| = 3. \end{cases}$$

In this case, one of the violations of *convexity* is that agent 1 has an inclusion cost of 7 for $\{1, 2\}$, while he has a lower inclusion cost of 4 for the larger group $\{1, 2, 3\}$. This means that whenever 1 has a valuation between 4 and 7, it may be optimal to exclude him from the smaller group by picking $\{2\}$ or it may be optimal to include him in the larger group by picking $\{1, 2, 3\}$. In particular, if 2 has a sufficiently high valuation, then the common member of both these groups must be included, and thus with his report agent 3 determines whether *both* he and 1 are included or excluded, regardless of what 1 reports between 4 and 7. We prove that it is this feature—that an agent has no voice even when changing his report *while* another controls their shared fate—that is incompatible with *equitable* mechanisms. Indeed, in this example, we construct (v_1, v_2) and (v_1', v_2) at which our axioms imply 3 must face the same *price* of winning. But this is impossible in a Groves mechanism, as the price must be precisely the externality imposed on the other agents by changing the social outcome, and thus for these profiles must vary with the valuation of agent 1. □

4. Discussion

4.1 *Costs*

Because convexity implies inclusion cost coverage, Theorem 1 and Theorem 2 together imply that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments. Remarkably, in this model, implementing production efficiency while providing equal opportunities never requires outside funding (such as subsidies from neighboring communities or borrowing from future generations) or redistribution through taxation. That said, this goal can only be achieved when production is both convex and symmetric.

We conclude this section by discussing some implications of our results in various settings.

Application 1 (Auctions). As discussed in Section 2, the problem of allocating k identical objects among agents with unit demand is modeled by a cost function that (i) assigns zero to any group with at most k members, and (ii) assigns infinity to any group with more than k members. Such a cost function is both *convex* and *symmetric*, so by Theorem 1 and Theorem 2, the exclusion pivot mechanism is a Groves mechanism that satisfies *no-deficit*, *voluntary participation*, and *no-envy*. Indeed, this is simply the Vickrey mechanism, and it is well known that it satisfies these properties (Vickrey (1961)).

Application 2 (Public goods). As discussed in Section 2, the problem of providing a pure public good is modeled by a cost function that (i) assigns some $\kappa \in [0, \infty)$ to serving everybody, (ii) assigns zero to serving nobody, and (iii) assigns infinity to serving any strict subset of the agents that is nonempty. Such a cost function violates *inclusion cost coverage*, so by Theorem 1 and Theorem 2, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. Indeed, this is a well-known formalization of the classic free-rider problem (Green and Laffont (1979)). As a particular example, the Lindahl mechanism does not satisfy either collection of axioms because it is not *strategy-proof*.

Application 3 (Natural monopolies). A cost function is *subadditive* if and only if for each disjoint pair $W, W' \subseteq N$, we have $C(W \cup W') \leq C(W) + C(W')$. The cost function is moreover *strictly subadditive* if and only if the inequality is strict for some pair of groups. It has been argued that these are the technologies that lead naturally to monopolies: if one firm produces W and another produces W', then the two firms can increase profits by merging (Baumol (1977)). Because these cost functions violate *inclusion cost coverage*, thus by Theorem 1 and Theorem 2, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. As discussed in Section 1, this strengthens a previous impossibility for the subclass of subadditive cost functions that are monotone and concave but not convex (Moulin and Shenker (2001)).

Application 4 (One-sided collusion in two-sided markets). Consider an economy with money and indivisible objects, where each individual desires at most one object and has preferences summarized by his valuation. Suppose that there are *n buyers*, each endowed with no object, and *m sellers*, each endowed with one object. In such an economy, there are Walrasian price vectors (Koopmans and Beckmann (1957); Gale (1960)), and moreover these vectors form a bounded lattice (Shapley and Shubik (1972)). It is well known that each *minimum-price Walrasian mechanism*, which uses reported buyer valuations to (i) calculate the minimum Walrasian price vector, and (ii) assign the buyers what they would receive in an associated Walrasian outcome, is *strategy-proof* (Demange (1982); Leonard (1983)).

Our analysis applies to the special case where the objects are identical; the classic example is a market for horses (Böhm-Bawerk (1888)). Indeed, suppose that the n buyer valuations are private information while the m seller valuations are common knowledge, and suppose that the buyers wish to coordinate on an outcome by means of a mechanism. If the seller valuations are ordered $\mathsf{S}_1 \leq \mathsf{S}_2 \leq \cdots \leq \mathsf{S}_m$, then feasibility is modeled by the cost function that (i) assigns zero to the empty set, (ii) assigns $\sum_{q'=1}^{q} \mathsf{S}_{q'}$ to any group of size $q \leq m$, and (iii) assigns infinity to any group with more than m members. This cost function is both *convex* and *symmetric*, so by Theorem 1 and Theorem 2, the exclusion pivot mechanisms are Groves mechanisms that satisfy *no-deficit*, *voluntary participation*, and *no-envy*. Indeed, in this case each exclusion pivot mechanism is a minimum-price Walrasian mechanism.

Application 5 (Minimum cost spanning trees). Suppose there is a *source* denoted 0, an *edge* is a distinct pair $i, j \in N \cup \{0\}$, and for each edge ij there is an associated cost $c_{ij} \in \mathbb{R}_+$. Moreover, for each $W \subseteq N$,

- a *spanning tree (for W)* is a collection of edges such that for each $i \in W$, there is a unique path from 0 to i;
- the cost of a spanning tree is the sum of the costs of its edges; and
- $C(W)$ is the minimum cost of a spanning tree.

Such *minimum cost spanning tree* cost functions (Claus and Kleitman (1973)) have been used to model the provision of utilities (such as electricity, water, sewage, and gas) and multicast transmissions.

Since Bird (1976), the literature has largely focused on the problem of sharing the cost $C(N)$ using cooperative game theory, implicitly requiring all agents to be connected. By contrast, our analysis applies when agents have finite valuations that are private information, in which case it may be inefficient to connect everybody. It is not difficult to see that, unless connecting everybody directly to the source always minimizes costs, the cost function does not satisfy *inclusion cost coverage*, so by Theorem 1 and Theorem 2, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*.

4.2 *Revenues*

When all costs are nonnegative, it is natural to imagine that $C(W)$ units of divisible money are used as an input, producing an indivisible good for members of W to consume. When all costs are nonpositive, however, it is more natural to imagine that members of W provide indivisible labor as an input, producing $-C(W)$ units of divisible money. In this case, we can interpret N as a set of workers and interpret the set of winners W as the group of workers that are employed.[7]

Indeed, for each environment (N, C), each valuation profile v, and each agent i, we can define the *revenue function* $R \equiv -C$ and the *reservation wage* $r_i \equiv -v_i$; observe that if i is willing to pay v_i to work, then he is willing to receive r_i to work. This alternative notation allows us to rewrite our axioms—for example, *production efficiency* requires that the group of winners maximizes the difference between revenue and the sum of the reservation wages, and *no-deficit* requires that the sum of the transfers to the agents is no greater than the produced revenue. As this change of notation has no impact on the logic of our arguments, our results can be applied to labor market environments that are specified by revenue (or "gross product") functions. To avoid duplicating all of our definitions with this alternative notation, we simply illustrate this point using the *gross substitutes* condition of Kelso and Crawford (1982).

Definition. Fix a set of agents $N = \{1, 2, \dots, n\}$. A *revenue function* is a mapping $R : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $R(\emptyset) = 0$. In the context of a revenue function, we let $\mathcal{W}$ denote the collection of groups with finite revenue. For each salary vector $s \in \mathbb{R}^N$, we let $\mathsf{D}(s)$ denote the firm's demand set, $\arg\max_{W \in \mathcal{W}} R(W) - \sum_{i \in W} s_i$. We say that R satisfies

[7] We thank an anonymous referee for suggesting this application.

- *gross substitutes* if and only if for each pair $s, s' \in \mathbb{R}^N$ such that for each $i \in N$ we have $s'_i \geq s_i$, and for each $W \in \mathsf{D}(s)$, there is a superset of $\{i \in W | s_i = s'_i\}$ in $\mathsf{D}(s')$; and
- *concavity* if and only if for each $i \in N$ and each pair $W, W' \subseteq N$ such that $\{i\} \subseteq W \subseteq W'$ and $W' \in \mathcal{W}$, we have
  - (i) $W \in \mathcal{W}$, and
  - (ii) $R(W') - R(W' \backslash \{i\}) \leq R(W) - R(W \backslash \{i\})$.

If a revenue function satisfies *gross substitutes*, then it satisfies *concavity*.[8] That said, the two conditions are not equivalent: Kelso and Crawford (1982) provide a revenue function that satisfies *concavity* but violates *gross substitutes*.

To simplify the interpretation, suppose that there is a single firm, and suppose that $R(W)$ specifies the revenue generated by this firm when it hires the members of W. If R satisfies *gross substitutes*, if the firm makes offers to workers who either tentatively accept or reject these offers, and if this is done iteratively in accordance with the *salary-adjustment process*, then this process converges to a discrete generalization of a competitive equilibrium in a finite number of steps (Kelso and Crawford (1982)). We are interested in comparing this dynamic procedure to a static alternative, where workers report their reservation wages to the firm, which then uses these reports to determine whom to hire and how much to pay each worker.

In fact, it follows immediately from our results that if the revenue function satisfies *gross substitutes*, then (i) each exclusion pivot mechanism is a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and (ii) there is an *envy-free* Groves mechanism if and only if workers are alike in production in that R is a symmetric function. Indeed, if R satisfies *gross substitutes*, then R is *concave*, so the associated cost function $C = -R$ is *convex*, so C satisfies *inclusion cost coverage*; the conclusions then follow from Theorem 1 and Theorem 2. On the other hand, if there are *envy-free* Groves mechanisms, then the revenue function satisfies *gross substitutes*: by Theorem 2, we have that C satisfies *symmetry* and *convexity*, so R satisfies *symmetry* and *concavity*, so R satisfies *gross substitutes* (Kelso and Crawford (1982); Theorem 6).[9] Finally, if the revenue function satisfies *concavity* but not *gross substitutes*, then the dynamic procedure need not work, but the exclusion pivot mechanism satisfies all of our axioms except *no-envy*.

[8] This implication was first established under *monotonicity* and the requirement that all groups have finite revenue (Gul and Stacchetti (1999); Lemma 5), and has since been shown to hold even without *monotonicity* (Paes Leme (2017); Theorem 2.2). In fact, this implication holds in general for our model; we briefly sketch the straightforward argument. First, argue that *gross substitutes* implies that if W belongs to $\mathcal{W}$, then all of its subsets belong to $\mathcal{W}$. Second, use the argument of Paes Leme (2017) to show that *gross substitutes* implies that for each distinct pair $i, j \in N$ and each $W \subseteq N \backslash \{i, j\}$ such that $W \cup \{i, j\} \in \mathcal{W}$, we have $R(W \cup \{i, j\}) - R(W \cup \{i\}) \leq R(W \cup \{j\}) - R(W)$. Finally, show that this version of submodularity implies *concavity*.

[9] Kelso and Crawford (1982) consider revenue functions for which all groups have finite revenue, but their proof extends directly to our setting.

4.3 *Preferences*

For our main results, we assume that each agent has quasi-linear preferences, and that moreover the set of admissible valuations is $\mathbb{R}$. How sensitive are our results to these assumptions? We consider both smaller preference domains and larger preference domains.[10]

First, for many economic applications, it is common knowledge that inclusion is a good that is costly to provide; thus in the working paper version (Mackenzie and Trudeau (2022)), we provide analogues to our main results that apply when (i) each agent's set of admissible valuations is $\mathbb{R}_+$, and (ii) each group is necessarily associated with a nonnegative cost. To state our results, we first introduce the modified cost function that associates each group with the cost its members would face if they were free to add any outsiders:

$$C^{\supseteq}(W) = \min_{\{W' \subseteq N | W' \supseteq W\}} C(W').$$

In the working paper version (Mackenzie and Trudeau (2022)), we prove that for nonnegative valuations and nonnegative costs, (i) there are autonomous mechanisms if and only if each *standard* pivot mechanism is autonomous if and only if $C^{\supseteq}$ satisfies inclusion cost coverage (Theorem 3), and (ii) there are equitable mechanisms if and only if each *standard* pivot mechanism is equitable if and only if $C^{\supseteq}$ is convex and symmetric (Theorem 4). It follows that when costs are moreover monotonic, our main results apply without modification.

Second, suppose we allow for income effects by relaxing quasi-linearity. In particular, suppose we assume only that for each agent i, (i) there is a *willingness to pay* w_i such that he is indifferent between $(0, 0)$ and $(-w_i, 1)$, and (ii) more money is better. It is clear that whenever there is no desirable mechanism with quasi-linearity, there is no desirable mechanism with this larger preference domain: otherwise the restriction to quasi-linear profiles would be desirable. On the other hand, whenever there are desirable quasi-linear mechanisms, one of them is an exclusion pivot mechanism; in this case, we can simply ask each agent to report his willingness to pay and use the exclusion pivot mechanism treating the reports as quasi-linear valuations.[11] With the exception of *production efficiency*, it is not hard to see that this extended mechanism inherits all axioms satisfied by its quasi-linear restriction: *no-envy* follows from the fact that losers always receive zero transfer, while the other axioms are trivial. The catch is that without quasi-linearity, maximizing the sum of the willingness to pay reports minus the cost does not imply there is no Pareto improvement that burns the same amount of money; this offers an intriguing direction for future work.

[10]We thank two anonymous referees for suggesting that we consider these two directions.

[11]This method of extending mechanisms beyond quasi-linear domains using the fixed reference bundle $(0, 0)$ has been previously considered for auctions (Saitoh and Serizawa (2008); Sakai (2008)), while an interesting variant in which reference bundles differ across agents been previously considered for public goods (Hashimoto and Saitoh (2016)).

Appendix A

In this appendix, we prove Theorem 1. We begin by stating (a corollary of a modification of) a fundamental theorem.

Definition. Fix an environment. A mechanism (τ, φ) is a *Groves mechanism* if and only if (i) φ is surplus-maximizing, and (ii) for each $i \in N$, there is $\theta_i : V_{-i} \to \mathbb{R}$ such that for each $v \in V$,

$$\tau_i(v) = \left(\left[\max_{W \subseteq N} \sigma_v(W)\right] - \varphi_i(v)v_i\right) + \theta_i(v_{-i}).$$

Theorem H (Holmström (1979)). *Fix an environment. A mechanism is* production efficient *and* strategy-proof *if and only if it is a Groves mechanism.*[12]

Our first lemma states that each exclusion pivot mechanism satisfies all requirements to be *autonomous* except possibly *no-deficit*. This is essentially a corollary of Theorem H; we include the formal statement, together with its short proof, for completeness.

Lemma A.1. *Fix an environment. Each exclusion pivot mechanism satisfies* production efficiency*,* strategy-proofness*, and* voluntary participation.

Proof. Let (τ, φ) be an exclusion pivot mechanism. For $i \in N$ and each $v_{-i} \in V_{-i}$, we can define $\theta_i(v_{-i}) \equiv -[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W)]$; as $i \in N$ was arbitrary, thus by Theorem H, (τ, φ) is *production efficient* and *strategy-proof*.

To see that (τ, φ) satisfies *voluntary participation*, let $i \in N$ and let $v \in V$. Then

$$\begin{aligned} \max_{W \subseteq N} \sigma_v(W) &\geq \max_{W \subseteq N \setminus \{i\}} \sigma_v(W) \\ &= \max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W), \end{aligned}$$

so $\tau_i(v) \geq -\varphi_i(v)v_i$. It follows immediately that (τ, φ) satisfies *voluntary participation*. □

Our second lemma states that the existence of *autonomous* mechanisms can always be determined using the exclusion pivot mechanisms. We begin by assuming there is an *autonomous* mechanism (τ, φ); the proof then consists of two steps.

First, we show that the exclusion pivot mechanism with winner policy φ, $(\tau^{\mathsf{p}|\varphi}, \varphi)$, is *autonomous*. This is shown by contradiction: if not, then by the previous lemma $(\tau^{\mathsf{p}|\varphi}, \varphi)$ satisfies all requirements but *no-deficit*, so there is some profile v and some agent i such that i receives a higher transfer at v under $(\tau^{\mathsf{p}|\varphi}, \varphi)$ than under (τ, φ). But

[12]Though the original theorem of Holmström (1979) does not involve cost functions, the straightforward adaptation of its proof goes through line-by-line for the modified version with cost functions. Our statement is in fact a corollary of the modified theorem, which follows because the domain $\times_N \mathbb{R}$ is a topologically connected space.

then we can construct a profile (x_i, v_{-i}) where i loses and pays under (τ, φ), contradicting that this mechanism satisfies *voluntary participation*.

Second, we show that each exclusion pivot mechanism $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ is *autonomous*. By taking advantage of the structure of exclusion pivot transfer policies, we write the difference in net transfers at an arbitrary profile between $(\tau^{\mathsf{p}|\varphi}, \varphi)$ and $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ as the difference in the costs of the selected groups, from which we can conclude that neither mechanism runs a deficit at the given profile. By the previous lemma, we are done.

Lemma A.2. *Fix an environment. If there is an* autonomous *mechanism, then each exclusion pivot mechanism is* autonomous.

Proof. Assume (τ, φ) is *autonomous*. By Theorem H, since (τ, φ) satisfies *production efficiency* and *strategy-proofness*, thus for each $i \in N$, there is $\theta_i : V_{-i} \to \mathbb{R}$ such that for each $v \in V$,

$$\tau_i(v) = \Big(\Big[\max_{W\subseteq N} \sigma_v(W)\Big] - \varphi_i(v)v_i\Big) + \theta_i(v_{-i}).$$

Let $(\tau^{\mathsf{p}|\varphi}, \varphi)$ be the exclusion pivot mechanism whose winner policy is φ.

∘ Step 1. The mechanism $(\tau^{\mathsf{p}|\varphi}, \varphi)$ is *autonomous*.

Assume, by way of contradiction, that $(\tau^{\mathsf{p}|\varphi}, \varphi)$ is not *autonomous*. Then by Lemma A.1, $(\tau^{\mathsf{p}|\varphi}, \varphi)$ violates *no-deficit*.

Since (τ, φ) satisfies *no-deficit* and $(\tau^{\mathsf{p}|\varphi}, \varphi)$ does not, thus there is $v \in V$ such that

$$\begin{aligned}\sum \tau_i^{\mathsf{p}|\varphi}(v) &> -C\big(\varphi(v)\big)\\ &\geq \sum \tau_i(v).\end{aligned}$$

Then there is $i \in N$ such that

$$\begin{aligned}\Big(\Big[\max_{W\subseteq N} \sigma_v(W)\Big] - \varphi_i(v)v_i\Big) - \Big[\max_{W\subseteq N\setminus\{i\}} \sigma_{(0,v_{-i})}(W)\Big] &= \tau_i^{\mathsf{p}|\varphi}(v)\\ &> \tau_i(v)\\ &= \Big(\Big[\max_{W\subseteq N} \sigma_v(W)\Big] - \varphi_i(v)v_i\Big) + \theta_i(v_{-i}).\end{aligned}$$

It cannot be that $\varphi(v) \subseteq N\setminus\{i\}$; else as φ is surplus-maximizing, thus for each $W \subseteq N\setminus\{i\}$,

$$\begin{aligned}\sigma_{(0,v_{-i})}\big(\varphi(v)\big) &= \sigma_v\big(\varphi(v)\big)\\ &\geq \sigma_v(W)\\ &= \sigma_{(0,v_{-i})}(W),\end{aligned}$$

so $[\max_{W\subseteq N} \sigma_v(W)] = \sigma_v(\varphi(v)) = \sigma_{(0,v_{-i})}(\varphi(v)) = [\max_{W\subseteq N\setminus\{i\}} \sigma_{(0,v_{-i})}(W)]$, and thus $\tau_i(v) < \tau_i^{\mathsf{p}|\varphi}(v) = 0$, contradicting that (τ, φ) satisfies *voluntary participation*.

Thus $i \in \varphi(v)$. Define

$$\sigma^* \equiv \max_{\{W \subseteq N | i \in W\}} \sigma_{(0, v_{-i})}(W).$$

Since $i \in \varphi(v)$ and φ is surplus-maximizing, thus

$$\begin{aligned}\sigma^* &\geq \sigma_{(0, v_{-i})}\big(\varphi(v)\big) \\ &= \sigma_v\big(\varphi(v)\big) - v_i \\ &\geq \sigma_v(\emptyset) - v_i \\ &= -v_i,\end{aligned}$$

so $\sigma^* > -\infty$. Define $x_i \in V_i$ by $x_i \equiv -(\sigma^* + 1)$.

By construction, at profile (x_i, v_{-i}), each group that includes i receives negative surplus and thus cannot be chosen over $\emptyset$, so surplus is maximized at a subset of $N\backslash\{i\}$. But then since $\theta_i(v_{-i}) < -[\max_{W \subseteq N\backslash\{i\}} \sigma_{(0, v_{-i})}(W)]$, thus

$$\begin{aligned}\tau_i(x_i, v_{-i}) &= \Big(\Big[\max_{W \subseteq N} \sigma_{(x_i, v_{-i})}(W)\Big] - 0\Big) + \theta_i(v_{-i}) \\ &< \Big(\Big[\max_{W \subseteq N} \sigma_{(x_i, v_{-i})}(W)\Big] - 0\Big) - \Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(0, v_{-i})}(W)\Big] \\ &= \Big(\Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(x_i, v_{-i})}(W)\Big] - 0\Big) - \Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(x_i, v_{-i})}(W)\Big] \\ &= 0,\end{aligned}$$

contradicting that (τ, φ) satisfies *voluntary participation*.

◦ STEP 2. Each exclusion pivot mechanism is *autonomous*.

Let $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ be an exclusion pivot mechanism and let $v \in V$. Then the net transfer at v according to $(\tau^{\mathsf{p}|\varphi}, \varphi)$ can be written

$$\begin{aligned}\sum \tau_i^{\mathsf{p}|\varphi}(v) &= \Big[\sum \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] - \varphi_i(v) v_i\Big) - \Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(0, v_{-i})}(W)\Big]\Big] \\ &= \Big[\sum \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] - \Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(0, v_{-i})}(W)\Big]\Big)\Big] - \Big[\sum \varphi_i(v) v_i\Big].\end{aligned}$$

Similarly, the net transfer at v according to $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ can be written

$$\sum \tau_i^{\mathsf{p}}(v) = \Big[\sum \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] - \Big[\max_{W \subseteq N\backslash\{i\}} \sigma_{(0, v_{-i})}(W)\Big]\Big)\Big] - \Big[\sum \varphi_i^{\mathsf{p}}(v) v_i\Big].$$

The difference between these net transfers is therefore

$$\sum \tau_i^{\mathsf{p}|\varphi}(v) - \sum \tau_i^{\mathsf{p}}(v) = \sum \varphi_i^{\mathsf{p}}(v) v_i - \sum \varphi_i(v) v_i.$$

Since φ and φ^{p} are both surplus-maximizing, thus

$$\sum \varphi_i(v) v_i - C\big(\varphi(v)\big) = \sum \varphi_i^{\mathsf{p}}(v) v_i - C\big(\varphi^{\mathsf{p}}(v)\big),$$

so altogether the difference between net transfers is

$$\sum \tau_i^{\mathsf{p}|\varphi}(v) - \sum \tau_i^{\mathsf{p}}(v) = C\big(\varphi^{\mathsf{p}}(v)\big) - C\big(\varphi(v)\big).$$

As $(\tau^{\mathsf{p}|\varphi}, \varphi)$ satisfies *no-deficit*, thus

$$\begin{aligned}\sum \tau_i^{\mathsf{p}}\big(\varphi(v)\big) + C\big(\varphi^{\mathsf{p}}(v)\big) &= \sum \tau_i^{\mathsf{p}|\varphi}(v) + C\big(\varphi(v)\big) \\ &\leq 0,\end{aligned}$$

from which it follows that $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ runs no deficit at v. Since $v \in V$ was arbitrary, thus $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ satisfies *no-deficit*, so by Lemma A.1, $(\tau^{\mathsf{p}}, \varphi^{\mathsf{p}})$ is *autonomous*. □

Our third lemma states that *inclusion cost coverage* guarantees each exclusion pivot mechanism is *autonomous*. In the proof, we argue that if the cost function satisfies this condition, then at each profile, any exclusion pivot mechanism requires both that (i) each loser pays nothing, and (ii) each winner pays at least his inclusion cost; it then follows that there is no deficit, and by our first lemma we are done.

Lemma A.3. *Fix an environment. If the cost function satisfies* inclusion cost coverage*, then each exclusion pivot mechanism is* autonomous.

Proof. Fix an environment whose cost function satisfies *inclusion cost coverage*, and let (τ, φ) be an exclusion pivot mechanism. By Lemma A.1, we need only prove that (τ, φ) satisfies *no-deficit*. Let $v \in V$ and define $W^* \equiv \varphi(v)$. Since φ is surplus-maximizing, thus $W^* \in \mathcal{W}$.

For each $i \in N \backslash W^*$, since φ is surplus-maximizing, we have

$$\begin{aligned}\sigma_{(0,v_{-i})}(W^*) &= \sigma_v(W^*) \\ &= \max_{W \subseteq N \backslash \{i\}} \sigma_v(W) \\ &= \max_{W \subseteq N \backslash \{i\}} \sigma_{(0,v_{-i})}(W).\end{aligned}$$

Altogether, for each $i \in N \backslash W^*$, $\tau_i(v) = 0$.

Let $i \in W^*$. By *inclusion cost coverage*, $W^* \backslash \{i\} \in \mathcal{W}$; thus

$$\begin{aligned}\max_{W \subseteq N \backslash \{i\}} \sigma_{(0,v_{-i})}(W) &\geq \sigma_{(0,v_{-i})}\big(W^* \backslash \{i\}\big) \\ &= \sum_{W^* \backslash \{i\}} v_j - C\big(W^* \backslash \{i\}\big) \\ &= \big(\sigma_v(W^*) + C(W^*) - v_i\big) - C\big(W^* \backslash \{i\}\big) \\ &= \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] + C(W^*) - v_i - C\big(W^* \backslash \{i\}\big)\Big).\end{aligned}$$

As $\varphi_i(v) = 1$, thus

$$\begin{aligned}
\tau_i(v) &= \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] - v_i\Big) - \Big[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)\Big] \\
&\leq \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] - v_i\Big) - \Big(\Big[\max_{W \subseteq N} \sigma_v(W)\Big] + C(W^*) - v_i - C(W^* \setminus \{i\})\Big) \\
&= C(W^* \setminus \{i\}) - C(W^*) \\
&= -\mathcal{I}_i(W^*).
\end{aligned}$$

Altogether, for each $i \in W^*$, $\tau_i(v) \leq -\mathcal{I}_i(W^*)$.

If $W^* = \emptyset$, it is immediate that $\sum \tau_i(v) = 0 = -C(\varphi(v))$, so (τ, φ) does not run a deficit at v. If W^* is nonempty, then since $W^* \in \mathcal{W}$, thus by *inclusion cost coverage*, $\sum_{W^*} \mathcal{I}_i(W^*) \geq C(W^*)$, so

$$\begin{aligned}
\sum \tau_i(v) &= \sum_{W^*} \tau_i(v) + \sum_{N \setminus W^*} \tau_i(v) \\
&= \sum_{W^*} \tau_i(v) \\
&\leq \sum_{W^*} \big(-\mathcal{I}_i(W^*)\big) \\
&\leq -C(W^*) \\
&= -C\big(\varphi(v)\big),
\end{aligned}$$

so (τ, φ) does not run a deficit at v. Since $v \in V$ was arbitrary, we are done. □

Our fourth lemma states that *inclusion cost coverage* is necessary for the existence of *autonomous* mechanisms. We proceed by contradiction, assuming that *inclusion cost coverage* is violated but there is an *autonomous* mechanism. Then (i) there is a group with finite cost W^* where *inclusion cost coverage* is violated, and (ii) each exclusion pivot mechanism is *autonomous*. For each sufficiently high x, we introduce a profile v^x where members of W^* have valuation x and the others have valuation $-nx$; the proof then consists of four steps.

First, we show that at any profile v^x, nobody outside of W^* can win: such an agent's valuation is so low that it would be better to select the empty set. Second, we show that each agent in W^* must win: such an agent's valuation is so high that it offsets any cost savings associated with his exclusion.

For each agent i in W^*, we then introduce the group W_i, which (i) includes only members of W^* and does not include i, (ii) has finite cost, (iii) among such groups is largest, and (iv) among such groups has smallest cost. Third, we show that at any profile v^x, the peers of i maximize their surplus without i at W_i; the arguments are similar to those in the first two steps.

To conclude, we consider a profile v^x where, by the first two steps, W^* wins. We first use the structure of the exclusion pivot transfer policy to show that each loser pays nothing and each winner i pays an amount involving the costs of W^* and W_i. If for each winner i, $W_i = W^*\backslash\{i\}$, then each winner pays precisely his inclusion cost, which does not cover costs, contradicting *no-deficit*. If instead for some winner i, $W_i \neq W^*\backslash\{i\}$, then the net transfer at v^x is at least the sum of x and a fixed term; but then as x was any sufficiently high number, there is some profile v^{x^*} where the cost of W^* is not covered, again contradicting *no-deficit*.

Lemma A.4. *Fix an environment. If the cost function violates* inclusion cost coverage*, then there is no* autonomous *mechanism.*

Proof. Fix an environment and assume, by way of contradiction, that the cost function violates *inclusion cost coverage* and there is an *autonomous* mechanism. Let (τ, φ) be an exclusion pivot mechanism. By Lemma A.2, (τ, φ) is *autonomous*.

Since C violates *inclusion cost coverage*, thus there is nonempty $W^* \in \mathcal{W}$ such that either

(i) there is $i \in W^*$ such that $W^*\backslash\{i\} \notin \mathcal{W}$, or

(ii) $\sum_{i\in W^*} \mathcal{I}_i(W^*) < C(W^*)$.

Necessarily, $|W^*| \geq 2$.

Define the *maximum absolute finite cost*, $\kappa \in \mathbb{R}_+$, by

$$\kappa \equiv \max_{W\in\mathcal{W}} \big|C(W)\big|.$$

Since $\emptyset \in \mathcal{W}$, thus κ is well-defined. For each $x \geq 2\kappa + 1$, define $v^x \in V$ by

$$v_i^x \equiv \begin{cases} x, & i \in W^*, \\ -nx, & \text{else.} \end{cases}$$

∘ Step 1: For each $x \geq 2\kappa + 1$ and each $i \in N\backslash W^*$, $i \notin \varphi(v^x)$.

If $N\backslash W^* = \emptyset$, we are done, so assume $N\backslash W^* \neq \emptyset$. Let $x \geq 2\kappa + 1$, let $i \in N\backslash W^*$, and let $W' \subseteq N$ such that $i \in W'$. Then

$$\begin{aligned} \sigma_{v^x}\big(W'\big) &= \sum_{W'\cap W^*} v_j^x + \sum_{W'\backslash W^*} v_j^x - C\big(W'\big) \\ &\leq (n-1)x + v_i^x + \kappa \\ &= (n-1)x - nx + \kappa \\ &= \kappa - x \\ &< 0 \\ &= \sigma_{v^x}(\emptyset), \end{aligned}$$

so since φ is surplus-maximizing, thus $\varphi(v^x) \neq W'$. Since W' including i was arbitrary, thus $i \notin \varphi(v^x)$. Since $x \geq 2\kappa + 1$ and $i \in N\backslash W^*$ were arbitrary, we are done.

∘ Step 2: For each $x \geq 2\kappa + 1$, $\varphi(v^x) = W^*$.

Let $x \geq 2\kappa + 1$. By Step 1, $\varphi(v^x) \subseteq W^*$. Let $W' \subsetneq W^*$. Then

$$\begin{aligned}\sigma_{v^x}(W^*) - \sigma_{v^x}(W') &= \Big(\sum_{W^*} v_j^x - C(W^*)\Big) - \Big(\sum_{W'} v_j^x - C(W')\Big) \\ &\geq \big(|W^*|x - \kappa\big) - \big((|W^*| - 1)x + \kappa\big) \\ &= x - 2\kappa \\ &> 0.\end{aligned}$$

Since $W' \subsetneq W^*$ was arbitrary, and since φ is surplus-maximizing, thus $\varphi(v^x) = W^*$.

Before proceeding to the next step, we first introduce a family of sets $\{W_i\}_{i \in W^*}$. In particular, for each $i \in W^*$, let $W_i \subseteq W^* \backslash \{i\}$ be such that

- $W_i \in \mathcal{W}$,
- for each $W \subseteq W^* \backslash \{i\}$ such that $W \in \mathcal{W}$, $|W_i| \geq |W|$, and
- for each $W \subseteq W^* \backslash \{i\}$ such that $W \in \mathcal{W}$ and $|W_i| = |W|$, $C(W_i) \leq C(W)$.

Thus W_i is one of the largest subsets of $W^* \backslash \{i\}$ with finite cost, and among these sets its cost is smallest. Since $\emptyset \subseteq W^* \backslash \{i\}$ and $\emptyset \in \mathcal{W}$, thus we can indeed select such a set to be W_i.

∘ Step 3: For each $x \geq 2\kappa + 1$ and each $i \in W^*$, $[\max_{W \subseteq N \backslash \{i\}} \sigma_{(0, v_{-i}^x)}(W)] = \sigma_{(0, v_{-i}^x)}(W_i)$.

Let $x \geq 2\kappa + 1$ and let $i \in W^*$. Define $\sigma^* \equiv [\max_{W \subseteq N \backslash \{i\}} \sigma_{(0, v_{-i}^x)}(W)]$, and let $W_i^* \subseteq N \backslash \{i\}$ such that $\sigma_{(0, v_{-i}^x)}(W_i^*) = \sigma^*$. Since $\sigma^* \geq \sigma_{(0, v_{-i}^x)}(\emptyset) = 0$, thus $W_i^* \in \mathcal{W}$.

We claim that $W_i^* \subseteq W^*$. Indeed, assume, by way of contradiction, there is $j \in W_i^* \backslash W^*$. Then as $i \notin W_i^*$, we have

$$\begin{aligned}\sigma_{(0, v_{-i}^x)}(W_i^*) &= |W_i^* \cap W^*|x + \big(|W_i^* \backslash W^*|\big)(-nx) - C(W_i^*) \\ &\leq (n-1)x + (-nx) + \kappa \\ &= \kappa - x \\ &< 0 \\ &= \sigma_{(0, v_{-i}^x)}(\emptyset),\end{aligned}$$

contradicting that W_i^* maximizes surplus at $(0, v_{-i}^*)$ among subsets of $N \backslash \{i\}$. Thus $W_i^* \subseteq W^*$, as desired.

Since $i \notin W_i^*$, altogether we have that $W_i^* \subseteq W^* \backslash \{i\}$ and $W_i^* \in \mathcal{W}$. Define $\mathcal{W}_1 \equiv \{W \subseteq W^* \backslash \{i\} | W \in \mathcal{W}\}$. If there are $W, W' \in \mathcal{W}_1$ such that $|W| > |W'|$, then

$$\begin{aligned}\sigma_{(0, v_{-i}^x)}(W) - \sigma_{(0, v_{-i}^x)}(W') &= \big(|W|x - C(W)\big) - \big(|W'|x - C(W')\big) \\ &= \big(|W| - |W'|\big)x + \big(C(W') - C(W)\big)\end{aligned}$$

$$\geq x - 2\kappa$$
$$> 0;$$

thus W_i^* is a set with highest cardinality in $\mathcal{W}_1$.

Define $\mathcal{W}_2 \equiv \{W \in \mathcal{W}_1 \| W| = |W_i^*|\}$. Clearly, W_i^* is a set with smallest cost in $\mathcal{W}_2$.

Define $\mathcal{W}_3 \equiv \{W \in \mathcal{W}_2 | C(W) = C(W_i^*)\}$. By construction, $W_i, W_i^* \in \mathcal{W}_3$. Clearly, $\sigma^* = \sigma_{(0, v_{-i}^x)}(W_i^*) = \sigma_{(0, v_{-i}^x)}(W_i)$, as desired.

∘ Step 4: Conclude.

Let $x \geq 2\kappa + 1$. For each $i \in N \backslash W^*$, by the fact that φ is surplus-maximizing and Step 2,

$$\begin{aligned}
\tau_i(v^x) &= \left(\left[\max_{W \subseteq N} \sigma_{v^x}(W)\right] - \varphi_i(v^x) v_i^x\right) - \left[\max_{W \subseteq N \backslash \{i\}} \sigma_{(0, v_{-i}^x)}(W)\right] \\
&= \left(\sigma_{v^x}(W^*) - 0\right) - \sigma_{v^x}(W^*) \\
&= 0.
\end{aligned}$$

For each $i \in W^*$, by Step 2, the fact that φ is surplus-maximizing, and Step 3,

$$\begin{aligned}
\tau_i(v^x) &= \left(\left[\max_{W \subseteq N} \sigma_{v^x}(W)\right] - \varphi_i(v^x) v_i^x\right) - \left[\max_{W \subseteq N \backslash \{i\}} \sigma_{(0, v_{-i}^x)}(W)\right] \\
&= \left(\left[\sigma_{v^x}(W^*)\right] - x\right) - \left[\sigma_{(0, v_{-i}^x)}(W_i)\right] \\
&= \left(\left[|W^*| x - C(W^*)\right] - x\right) - \left[|W_i| x - C(W_i)\right] \\
&= \left(|W^*| - |W_i| - 1\right) x + \left(C(W_i) - C(W^*)\right).
\end{aligned}$$

Thus

$$\sum \tau_i(v^x) = \left[\sum_{W^*} \left(|W^*| - |W_i| - 1\right) x\right] + \left[\sum_{W^*} C(W_i) - C(W^*)\right].$$

For each $i \in W^*$, since $W_i \subseteq W^* \backslash \{i\}$, thus $|W_i| \leq |W^*| - 1$.

To conclude, we consider two cases: (i) for each $i \in W^*$, $|W_i| = |W^*| - 1$, and (ii) there is $i \in W^*$ such that $|W_i| < |W^*| - 1$. Equivalently, these two cases are (i) for each $i \in W^*$, $W^* \backslash \{i\}$ has finite cost, and (ii) there is $i \in W^*$ such that $W^* \backslash \{i\}$ has infinite cost.

First, assume that for each $i \in W^*$, $|W_i| = |W^*| - 1$. Then for each $i \in W^*$, $W_i = W^* \backslash \{i\}$, and thus $W^* \backslash \{i\} \in \mathcal{W}$. Then $\sum_{i \in W^*} \mathcal{I}_i(W^*) < C(W^*)$, so by construction of W^* and Step 2,

$$\begin{aligned}
\sum \tau_i(v^x) &= \left[\sum_{W^*} C(W_i) - C(W^*)\right] \\
&= \left[\sum_{W^*} C(W^* \backslash \{i\}) - C(W^*)\right] \\
&= \sum_{W^*} \left(-\mathcal{I}_i(W^*)\right)
\end{aligned}$$

$$> -C(W^*)$$
$$= -C(\varphi(v^x)),$$

contradicting *no-deficit*.

Second, assume there is $i \in W^*$ such that $|W_i| < |W^*| - 1$. Then

$$\sum \tau_i(v^x) \geq x + \left[\sum_{W^*} C(W_i) - C(W^*)\right].$$

Since $x \geq 2\kappa + 1$ was arbitrary, thus using Step 2, there is a sufficiently high $x^* \geq 2\kappa + 1$ such that

$$\sum \tau_i(v^{x^*}) > -C(W^*)$$
$$= -C(\varphi(v^{x^*})),$$

contradicting *no-deficit*. □

To conclude, Theorem 1 is a direct corollary of our previous lemmas.

Theorem 1 (Repeated). *Fix an environment. The following are equivalent:*

- *there is a mechanism that satisfies* production efficiency, strategy-proofness, no-deficit, *and* voluntary participation*;*
- *each exclusion pivot mechanism satisfies* production efficiency, strategy-proofness, no-deficit, *and* voluntary participation*; and*
- *the cost function satisfies* inclusion cost coverage.

Proof. The first item implies the third by Lemma A.4, the third implies the second by Lemma A.3, and the second implies the first trivially. □

Appendix B

In this appendix, we prove Theorem 2. We begin by proving that if costs are not *convex*, then there are no *equitable* mechanisms. We proceed by contradiction, assuming that costs are not *convex* but there is an *equitable* mechanism; the proof then consists of four steps.

First, we use the lack of *convexity* to identify an agent i and a pair of groups W_-, W_+ that together satisfy four properties. These properties imply that at certain profiles, another agent can determine whether $W_- \backslash \{i\}$ wins or W_+ wins, and thus determine whether or not i wins. In particular, these four properties are: (i) i belongs to W_-, which is contained in W_+; (ii) the inclusion cost for i in W_+ is smaller than his inclusion cost in W_-; (iii) $W_- \backslash \{i\}$ certainly has finite cost (and by the previous property, W_+ does as well); and (iv) among groups that contain $W_- \backslash \{i\}$ and that are contained in W_+, the only that might have finite cost are $W_- \backslash \{i\}$, W_-, $W_+ \backslash \{i\}$, and W_+.

Next, we take the other agent j to be anybody in W_+ but not W_-. In the second step, we carefully define (i) high and low valuations for i, and (ii) high and low valuations for j. We then use these to construct four profiles: v^{++}, v^{+-}, v^{-+}, and v^{--}. At profiles where the first superscript is $+$, i has his high valuation; at profiles where the first superscript is $-$, i has his low valuation; the second superscript plays the same role for j. The valuations of agents in $N \backslash \{i, j\}$ are fixed so that across these four profiles, *production efficiency* implies that the winning group must contain $W_- \backslash \{i\}$ and be contained in W_+. Both valuations of i are between his inclusion cost in W_+ and his inclusion cost in W_-, while the valuations of j allow him to demand inclusion or exclusion across these profiles.

In the third step, we show that across the four profiles, W_+ wins whenever j has his high valuation and $W_- \backslash \{i\}$ wins whenever j has his low valuation. In the final step, we conclude by deriving a contradiction: j must receive the same transfer at both profiles where he wins and the same transfer at both profiles where he loses, which is impossible for any Groves mechanism.

Lemma B.1. *Fix an environment. If the cost function violates* convexity*, then there are no* equitable *mechanisms.*

Proof. Let C be a cost function that violates *convexity*, and assume, by way of contradiction, that (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

◦ Step 1: There are $i \in N$ and $W_-, W_+ \subseteq N$ such that

(i) $\{i\} \subseteq W_- \subseteq W_+$;

(ii) $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$;

(iii) $W_- \backslash \{i\} \in \mathcal{W}$; and

(iv) for each $W \in 2^N \backslash \{W_- \backslash \{i\}, W_-, W_+ \backslash \{i\}, W_+\}$ such that $W_- \backslash \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$.

Since C violates *convexity*, there are $k \in N$ and $W'_-, W'_+ \subseteq N$ such that (i) $\{k\} \subseteq W'_- \subseteq W'_+$, and (ii) $\mathcal{I}_k(W'_-) > \mathcal{I}_k(W'_+)$.

We first claim that there are $i \in N$ and $W^*_- \subseteq W'_-$ such that (i) $\{i\} \subseteq W^*_- \subseteq W'_+$, (ii) $\mathcal{I}_i(W^*_-) > \mathcal{I}_i(W'_+)$, and (iii) $W^*_- \backslash \{i\} \in \mathcal{W}$. Indeed, if $W'_- \backslash \{k\} \in \mathcal{W}$, simply define $i \equiv k$ and $W^*_- \equiv W'_-$. If $W'_- \backslash \{k\} \notin \mathcal{W}$, then because $\mathcal{I}_k(W'_-) > \mathcal{I}_k(W'_+) \geq -\infty$, necessarily $W'_- \notin \mathcal{W}$. In this case, define $W_0 \equiv W'_-$, and for each $t \in \{1, 2, \ldots, |W'_-|\}$, define

- $i_t \equiv \min W_{t-1}$, and
- $W_t \equiv W_{t-1} \backslash \{i_t\}$.

It is straightforward to verify that $W_{|W'_-|} = \emptyset$, so $C(W_{|W'_-|}) = 0$. Thus there is $t \in \{1, 2, \ldots, |W'_-|\}$ such that $W_t \in \mathcal{W}$ and $W_{t-1} \notin \mathcal{W}$. Define $i \equiv i_t$ and define $W^*_- \equiv W_{t-1}$; then (i) $\{i\} \subseteq W^*_- \subseteq W'_- \subseteq W'_+$, and (iii) $W^*_- \backslash \{i\} = W_t \in \mathcal{W}$. Moreover, $W^*_- = W_{t-1} \notin \mathcal{W}$, so $\mathcal{I}_i(W^*_-) = \infty$. Since $\{i\} \subseteq W'_+$, thus we can define $\mathcal{I}_i(W'_+)$; and since $\mathcal{I}_k(W'_+) < \mathcal{I}_k(W'_-) \leq$

∞, thus $W'_+ \in \mathcal{W}$, so $\mathcal{I}_i(W'_+) < \infty$; altogether, then, we have (ii) $\mathcal{I}_i(W'_+) < \infty = \mathcal{I}_i(W^*_-)$. This completes the proof of the claim.

Define

$$\mathcal{W}_+ \equiv \{W \subseteq N | \{i\} \subseteq W^*_- \subseteq W \subseteq W'_+ \text{ and } \mathcal{I}_i(W^*_-) > \mathcal{I}_i(W)\},$$

which is nonempty as $W'_+ \in \mathcal{W}_+$. Let W_+ be a set that is minimal in $\mathcal{W}_+$ with respect to set inclusion.

Define

$$\mathcal{W}_- \equiv \{W \subseteq N | \{i\} \subseteq W^*_- \subseteq W \subseteq W_+, \mathcal{I}_i(W) > \mathcal{I}_i(W_+), \text{ and } W \backslash \{i\} \in \mathcal{W}\},$$

which is nonempty as $W^*_- \in \mathcal{W}_-$. Let W_- be a set that is maximal in $\mathcal{W}_-$ with respect to set inclusion.

By construction, we have that (i) $\{i\} \subseteq W_- \subseteq W_+$, (ii) $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, and (iii) $W_- \backslash \{i\} \in \mathcal{W}$. To complete the proof of Step 1, let $W \in 2^N$ such that $W_- \subsetneq W \subsetneq W_+$. Then we cannot have $\mathcal{I}_i(W) \leq \mathcal{I}_i(W_+)$, else $\mathcal{I}_i(W) < \mathcal{I}_i(W^*_-)$, so $W \in \mathcal{W}_+$, contradicting the minimality of W_+ in $\mathcal{W}_+$. Thus $\mathcal{I}_i(W) > \mathcal{I}_i(W_+)$, so we must have $W \backslash \{i\} \notin \mathcal{W}$; else we have $W \in \mathcal{W}_-$, contradicting the maximality of W_- in $\mathcal{W}_-$. As $W \backslash \{i\} \notin \mathcal{W}$, we must have $W \notin \mathcal{W}$; else $\mathcal{I}_i(W) = -\infty \leq \mathcal{I}_i(W_+) < \mathcal{I}_i(W^*_-)$, so $W \in \mathcal{W}_+$, again contradicting the minimality of W_+ in $\mathcal{W}_+$. Altogether, then, $W \backslash \{i\} \notin \mathcal{W}$ and $W \notin \mathcal{W}$. As $W \subseteq N$ such that $W_- \subsetneq W \subsetneq W_+$ was arbitrary, thus we have established (iv) for each $W \in 2^N \backslash \{W_- \backslash \{i\}, W_-, W_+ \backslash \{i\}, W_+\}$ such that $W_- \backslash \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$, as desired.

∘ Step 2: Define $j \in N$, define $\kappa \in \mathbb{R}_+$, define $v_i^+, v_i^-, v_j^+, v_j^-, x^+, x^- \in \mathbb{R}$ such that $x^+ > 0 > x^-$, and define $v^{++}, v^{+-}, v^{-+}, v^{--} \in V$.

Because $W_- \subseteq W_+$ and $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, thus $W_- \subsetneq W_+$. Define $j \equiv \min W_+ \backslash W_-$. Define the *maximum absolute finite cost*, $\kappa \in \mathbb{R}_+$, by

$$\kappa = \max_{W \in \mathcal{W}} |C(W)|.$$

As $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, choose $v_i^+, v_i^- \in \mathbb{R}$ such that $\mathcal{I}_i(W_-) > v_i^+ > v_i^- > \mathcal{I}_i(W_+)$.

Define $v_j^+, v_j^-, x^+, x^- \in \mathbb{R}$ such that $x^+ > 0 > x^-$ by

$$\begin{aligned}
v_j^+ &\equiv -v_i^- + 2\kappa + 1, \\
v_j^- &\equiv -v_i^+ - 2\kappa - 1, \\
x^+ &\equiv \max\{v_i^+, 0\} + \max\{v_j^+, 0\} + 2\kappa + 1, \quad \text{and} \\
x^- &\equiv \min\{v_i^-, 0\} + \min\{v_j^-, 0\} - 2\kappa - 1.
\end{aligned}$$

Finally, define $v^{++}, v^{+-}, v^{-+}, v^{--} \in V$ by

(i) for each $v \in \{v^{++}, v^{+-}\}$, $v_i \equiv v_i^+$,

(ii) for each $v \in \{v^{-+}, v^{--}\}$, $v_i \equiv v_i^-$,

(iii) for each $v \in \{v^{++}, v^{-+}\}$, $v_j \equiv v_j^+$,

(iv) for each $v \in \{v^{+-}, v^{--}\}$, $v_j \equiv v_j^-$, and

(v) for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$ and each $k \in N \backslash \{i, j\}$, we have

$$v_k \equiv \begin{cases} x^+, & k \in W_- \backslash \{i\}, \\ x^-, & k \in N \backslash W_+, \\ 0, & \text{else.} \end{cases}$$

∘ Step 3: For each $v \in \{v^{++}, v^{-+}\}$, $\varphi(v) = W_+$, and for each $v \in \{v^{+-}, v^{--}\}$, $\varphi(v) = W_- \backslash \{i\}$.

This proof of this step consists of two claims, then a conclusion. First, we claim that for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$, $W_- \backslash \{i\} \subseteq \varphi(v) \subseteq W_+$. Indeed, let $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$. To see that $W_- \backslash \{i\} \subseteq \varphi(v)$, let $W \subseteq N$ such that $W_- \backslash \{i\} \nsubseteq W$. Then there is $k^* \in (W_- \backslash \{i\}) \backslash W$, so

$$\begin{aligned} \sigma_v(W_- \backslash \{i\}) - \sigma_v(W) &= \left[\sum_{W_- \backslash \{i\}} v_k - C(W_- \backslash \{i\})\right] - \left[\sum_W v_k - C(W)\right] \\ &\geq \left[|W_- \backslash \{i\}| x^+ - \kappa\right] \\ &\quad - \left[(|W_- \backslash \{i\}| - 1)x^+ + \max\{v_i^+, 0\} + \max\{v_j^+, 0\} + \kappa\right] \\ &= x^+ - 2\kappa - \max\{v_i^+, 0\} - \max\{v_j^+, 0\} \\ &> 0, \end{aligned}$$

so as φ is *surplus-maximizing* we have $\varphi(v) \neq W$. Since $W \subseteq N$ such that $W_- \backslash \{i\} \nsubseteq W$ was arbitrary, thus $W_- \backslash \{i\} \subseteq \varphi(v)$, as desired. To see that $\varphi(v) \subseteq W_+$, let $W \subseteq N$ such that $W \nsubseteq W_+$. Then there is $k^* \in W \backslash W_+$, so

$$\begin{aligned} \sigma_v(W_+) - \sigma_v(W) &= \left[\sum_{W_+} v_k - C(W_+)\right] - \left[\sum_W v_k - C(W)\right] \\ &\geq \left[(|W_-| - 1)x^+ + v_i + v_j - \kappa\right] \\ &\quad - \left[x^- + (|W_-| - 1)x^+ + \max\{v_i, 0\} + \max\{v_j, 0\} + \kappa\right] \\ &= -x^- + (v_i - \max\{v_i, 0\}) + (v_j - \max\{v_j, 0\}) - 2\kappa \\ &= -x^- + \min\{v_i, 0\} + \min\{v_j, 0\} - 2\kappa \\ &\geq -x^- + \min\{v_i^-, 0\} + \min\{v_j^-, 0\} - 2\kappa \\ &> 0, \end{aligned}$$

so as φ is *surplus-maximizing* we have $\varphi(v) \neq W$. Since $W \subseteq N$ such that $W \subsetneq W_+$ was arbitrary, thus $\varphi(v) \subseteq W_+$, as desired. This completes the proof of our first claim.

Second, we claim that for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$, $\varphi(v) \in \{W_- \backslash \{i\}, W_+\}$. Indeed, let $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$. By the first claim, $W_- \backslash \{i\} \subseteq \varphi(v) \subseteq W_+$. By Step 1, for each $W \in 2^N \backslash \{W_- \backslash \{i\}, W_-, W_+ \backslash \{i\}, W_+\}$ such that $W_- \backslash \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$, so $\sigma_v(W) = -\infty < 0 = \sigma_v(\emptyset)$, so as φ is *surplus-maximizing*, we have $\varphi(v) \neq W$. Altogether, then, $\varphi(v) \in \{W_- \backslash \{i\}, W_-, W_+ \backslash \{i\}, W_+\}$. Because $\mathcal{I}_i(W_-) > v_i > \mathcal{I}_i(W_+)$, it is straightforward to verify that $\sigma_v(W_- \backslash \{i\}) > \sigma_v(W_-)$ and $\sigma_v(W_+) > \sigma_v(W_+ \backslash \{i\})$; thus as φ is *surplus-maximizing*, we have $\varphi(v) \notin \{W_-, W_+ \backslash \{i\}\}$. This completes the proof of our second claim.

To conclude, first let $v \in \{v^{++}, v^{-+}\}$. Then

$$\begin{aligned}
\sigma_v(W_+) - \sigma_v\big(W_- \backslash \{i\}\big) &= \Bigg[\sum_{W_+} v_k - C(W_+)\Bigg] - \Bigg[\sum_{W_- \backslash \{i\}} v_k - C\big(W_- \backslash \{i\}\big)\Bigg] \\
&= \Bigg[\sum_{W_- \backslash \{i\}} v_k + v_i + v_j^+ - C(W_+)\Bigg] - \Bigg[\sum_{W_- \backslash \{i\}} v_k - C\big(W_- \backslash \{i\}\big)\Bigg] \\
&\geq \Bigg[\sum_{W_- \backslash \{i\}} v_k + v_i + v_j^+ - \kappa\Bigg] - \Bigg[\sum_{W_- \backslash \{i\}} v_k + \kappa\Bigg] \\
&= v_i + v_j^+ - 2\kappa \\
&\geq v_i^- + v_j^+ - 2\kappa \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing*, $\varphi(v) \neq W_- \backslash \{i\}$. By the second claim, $\varphi(v) = W_+$. Finally, let $v \in \{v^{+-}, v^{--}\}$. Then

$$\begin{aligned}
\sigma_v\big(W_- \backslash \{i\}\big) - \sigma_v(W_+) &= \Bigg[\sum_{W_- \backslash \{i\}} v_k - C\big(W_- \backslash \{i\}\big)\Bigg] - \Bigg[\sum_{W_+} v_k - C(W_+)\Bigg] \\
&= \Bigg[\sum_{W_- \backslash \{i\}} v_k - C\big(W_- \backslash \{i\}\big)\Bigg] - \Bigg[\sum_{W_- \backslash \{i\}} v_k + v_i + v_j^- - C(W_+)\Bigg] \\
&\geq \Bigg[\sum_{W_- \backslash \{i\}} v_k - \kappa\Bigg] - \Bigg[\sum_{W_- \backslash \{i\}} v_k + v_i + v_j^- + \kappa\Bigg] \\
&= -2\kappa - v_i - v_j^- \\
&\geq -2\kappa - v_i^+ - v_j^- \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing*, $\varphi(v) \neq W_+$. By the second claim, $\varphi(v) = W_- \backslash \{i\}$. This completes the proof of the step.

∘ Step 4: Conclude.

By Step 3, we have that both i and j win at v^{++} and v^{-+}, and we have that both i and j lose at v^{+-} and v^{--}.

By *no-envy*, $\tau_j(v^{++}) = \tau_i(v^{++})$. By *strategy-proofness*, $\tau_i(v^{++}) = \tau_i(v^{-+})$. By *no-envy*, $\tau_i(v^{-+}) = \tau_j(v^{-+})$. Altogether, $\tau_j(v^{++}) = \tau_j(v^{-+})$.

By *no-envy*, $\tau_j(v^{+-}) = \tau_i(v^{+-})$. By *strategy-proofness*, $\tau_i(v^{+-}) = \tau_i(v^{--})$. By *no-envy*, $\tau_i(v^{--}) = \tau_j(v^{--})$. Altogether, $\tau_j(v^{+-}) = \tau_j(v^{--})$.

As $v_i^+ > v_i^-$, thus by Step 3 and Theorem H, we have

$$\begin{aligned}\tau_j(v^{++}) - \tau_j(v^{+-}) &= \sigma_{v^{++}}(W_+) - v_j^+ - \sigma_{v^{+-}}(W_-\backslash\{i\}) \\ &> \sigma_{v^{-+}}(W_+) - v_j^+ - \sigma_{v^{--}}(W_-\backslash\{i\}) \\ &= \tau_j(v^{-+}) - \tau_j(v^{--}).\end{aligned}$$

But then

$$\begin{aligned}0 &= \tau_j(v^{++}) - \tau_j(v^{-+}) \\ &> \tau_j(v^{+-}) - \tau_j(v^{--}) \\ &= 0,\end{aligned}$$

contradicting that $0 = 0$. □

Our second impossibility lemma in this appendix states that if costs are *convex* but not *symmetric*, then there are no *equitable* mechanisms. In the proof, we first use the absence of *symmetry* to identify a group W^* and a pair of agents i, j outside of that group such that the cost of $W^* \cup \{i\}$ is less than the cost of $W^* \cup \{j\}$. We then use *convexity* to construct a particular nondegenerate interval from which we pick a low valuation for i and a high valuation for j; we complete the profile v^* by assigning a large positive valuation to members of W^* and a large negative valuation to the others. By *production efficiency*, $W^* \cup \{i\}$ must win at this profile, but *no-envy* then implies that the valuation of i is at least the valuation of j, which is not true.

Lemma B.2. *Fix an environment. If the cost function satisfies* convexity *but not* symmetry*, then there are no* equitable *mechanisms.*

Proof. Fix an environment whose cost function satisfies *convexity* but not *symmetry*, and assume, by way of contradiction, that (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

We claim there are $W^* \subseteq N$ and $i, j \in N\backslash W^*$ such that $C(W^* \cup \{i\}) < C(W^* \cup \{j\})$. Indeed, since C is not *symmetric*, there are $W, W' \subseteq N$ such that $|W| = |W'|$ and $C(W) < C(W')$. Define $W_0 \equiv W$, and for each $t \in \{1, 2, \dots, |W\backslash W'|\}$, define

- $i_t \equiv \min(W_{t-1}\backslash W')$,
- $j_t \equiv \min(W'\backslash W_{t-1})$, and
- $W_t \equiv (W_{t-1}\backslash\{i_t\}) \cup \{j_t\}$.

It is straightforward to verify that $W_{|W\backslash W'|} = W'$. Thus there is $t \in \{1, 2, \dots, |W\backslash W'|\}$ such that $C(W_{t-1}) < C(W_t)$. Define $W^* \equiv W_t \cap W_{t-1}$, define $i \equiv i_t$, and define $j \equiv j_t$; then $W^* \subseteq N$ and $i, j \in N\backslash W^*$ are as desired.

Since $W^* \cup \{i\} \in \mathcal{W}$, thus $W^* \in \mathcal{W}$; else there is $k \in W^*$ such that $\mathcal{I}_k(W^*) = \infty > \mathcal{I}_k(W^* \cup \{i\})$, contradicting *convexity*. Then $\mathcal{I}_j(W^* \cup \{j\}) \neq -\infty$, so we can define the *maximum absolute finite cost* $\kappa \in \mathbb{R}_+$, the *upper bound* $v^{\max} \in \mathbb{R}$, and the *lower bound* $v^{\min} \in \mathbb{R}$, by

$$\begin{aligned}\kappa &\equiv \max_{W \in \mathcal{W}} \big|C(W)\big|,\\ v^{\max} &\equiv \min\big\{2\kappa + 1, \mathcal{I}_j\big(W^* \cup \{j\}\big)\big\}, \quad \text{and}\\ v^{\min} &\equiv \mathcal{I}_i\big(W^* \cup \{i\}\big).\end{aligned}$$

We claim $v^{\max} > v^{\min}$. Indeed, since $W^*, W^* \cup \{i\} \in \mathcal{W}$, thus it cannot be that $\mathcal{I}_i(W^* \cup \{i\}) > 2\kappa$, else

$$\begin{aligned}C\big(W^* \cup \{i\}\big) &= C\big(W^*\big) + \mathcal{I}_i\big(W^* \cup \{i\}\big)\\ &> -\kappa + 2\kappa,\end{aligned}$$

contradicting the construction of κ as the maximum absolute finite cost. Then $2\kappa + 1 > 2\kappa \geq \mathcal{I}_i(W^* \cup \{i\})$. Moreover, since $C(W^* \cup \{j\}) > C(W^* \cup \{i\})$ and $W^* \in \mathcal{W}$, thus $\mathcal{I}_j(W^* \cup \{j\}) > \mathcal{I}_i(W^* \cup \{i\})$. Altogether, then, $v^{\max} > v^{\min}$, as desired.

Let $v^h, v^l \in \mathbb{R}$ such that $v^{\max} > v^h > v^l > v^{\min}$, and define $v^* \in V$ such that for each $k \in N$,

$$v_k^* \equiv \begin{cases} v^l, & k = i,\\ v^h, & k = j,\\ 6\kappa + 3, & k \in W^*,\\ -(6\kappa + 3), & \text{else.}\end{cases}$$

We first claim that $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$. Indeed, if there were $k^* \in W^* \backslash \varphi(v^*)$, then

$$\begin{aligned}\sigma_{v^*}\big(W^*\big) - \sigma_{v^*}\big(\varphi\big(v^*\big)\big) &= \left(\sum_{W^*} v_k^* - C\big(W^*\big)\right) - \left(\sum_{\varphi(v^*)} v_k^* - C\big(\varphi\big(v^*\big)\big)\right)\\ &\geq \big(\big|W^*\big|(6\kappa + 3) - \kappa\big)\\ &\quad - \big(\big(|W^*| - 1\big)\big(6\kappa + 3\big) + \max\big\{v^l, 0\big\} + \max\big\{v^h, 0\big\} + \kappa\big)\\ &= (6\kappa + 3) - 2\kappa - \max\big\{v^l, 0\big\} - \max\big\{v^h, 0\big\}\\ &> (6\kappa + 3) - 2\kappa - (2\kappa + 1) - (2\kappa + 1)\\ &> 0,\end{aligned}$$

contradicting that φ is *surplus-maximizing*. If there were $k^* \in \varphi(v^*) \backslash (W^* \cup \{i, j\})$, then

$$\sigma_{v^*}\big(W^*\big) - \sigma_{v^*}\big(\varphi\big(v^*\big)\big) = \left(\sum_{W^*} v_k^* - C\big(W^*\big)\right) - \left(\sum_{\varphi(v^*)} v_k^* - C\big(\varphi\big(v^*\big)\big)\right)$$

$$
\begin{aligned}
&\geq \big(|W^*|(6\kappa+3)-\kappa\big) \\
&\quad -\big(|W^*|(6\kappa+3)+\max\{v^l,0\}+\max\{v^h,0\}-(6\kappa+3)+\kappa\big) \\
&=(6\kappa+3)-2\kappa-\max\{v^l,0\}-\max\{v^h,0\} \\
&>(6\kappa+3)-2\kappa-(2\kappa+1)-(2\kappa+1) \\
&>0,
\end{aligned}
$$

contradicting that φ is *surplus-maximizing*. Thus $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$, as desired.

Next, we claim $\varphi(v^*) = W^* \cup \{i\}$. Indeed, if $\varphi(v^*) = W^* \cup \{i, j\}$, then by *production efficiency*, $W^* \cup \{i, j\} \in \mathcal{W}$. Since $W^* \cup \{i\} \in \mathcal{W}$, thus by *convexity*

$$
\begin{aligned}
\sigma_{v^*}\big(W^* \cup \{i\}\big)-\sigma_{v^*}\big(W^* \cup \{i,j\}\big) &= -v_j^* + \mathcal{I}_j\big(W^* \cup \{i,j\}\big) \\
&\geq -v^h + \mathcal{I}_j\big(W^* \cup \{j\}\big) \\
&> 0,
\end{aligned}
$$

contradicting that φ is *surplus-maximizing*. Similarly, if $\varphi(v^*) = W^* \cup \{j\}$, then by *production efficiency*, $W^* \cup \{j\} \in \mathcal{W}$. Since $W^* \in \mathcal{W}$, thus

$$
\begin{aligned}
\sigma_{v^*}\big(W^*\big)-\sigma_{v^*}\big(W^* \cup \{j\}\big) &= -v_j^* + \mathcal{I}_j\big(W^* \cup \{j\}\big) \\
&= -v^h + \mathcal{I}_j\big(W^* \cup \{j\}\big) \\
&> 0,
\end{aligned}
$$

contradicting that φ is *surplus-maximizing*. Finally, if $\varphi(v^*) = W^*$, then since $W^*, W^* \cup \{i\} \in \mathcal{W}$, thus

$$
\begin{aligned}
\sigma_{v^*}\big(W^* \cup \{i\}\big)-\sigma_{v^*}\big(W^*\big) &= v_i^* - \mathcal{I}_i\big(W^* \cup \{i\}\big) \\
&= v^l - \mathcal{I}_i\big(W^* \cup \{i\}\big) \\
&> 0,
\end{aligned}
$$

contradicting that φ is *surplus-maximizing*. As $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$, altogether, then, $\varphi(v^*) = W^* \cup \{i\}$, as desired.

To conclude, by *no-envy*, $\tau_i(v^*) + v^h \leq \tau_j(v^*)$ and $\tau_j(v^*) \leq \tau_i(v^*) + v^l$. But then

$$
\begin{aligned}
\tau_i\big(v^*\big) &\leq \tau_j\big(v^*\big) - v^h \\
&\leq \tau_i\big(v^*\big) + v^l - v^h,
\end{aligned}
$$

so $v^h \leq v^l$, contradicting that $v^h > v^l$. □

Finally, we prove that if costs are *convex* and *symmetric*, then each exclusion pivot mechanism is *equitable*. We begin by taking an arbitrary valuation profile and using it to construct a reported demand curve, which is used alongside the fixed supply curve given by the marginal costs. By *production efficiency*, winners must be served so long as demand exceeds supply, and may be served when the two are equal; we define $\check{q}$ to be the minimum efficient quantity and $\hat{q}$ to be the maximum efficient quantity. Similarly, for a given agent i, we define $\hat{q}_i$ to be the maximum efficient quantity given i is removed.

The key observation is that for all winners, $\hat{q}$ and $\hat{q}_i$ share the same relationship. Indeed, if the $(\hat{q}+1)$th-highest valuation is at least the $\hat{q}$th-highest marginal cost, then for each winner we have that $\hat{q}_i = \hat{q}$; otherwise, for each winner we have that $\hat{q}_i = \hat{q} - 1$. In the first case, any winner can be replaced when he is removed, in the sense that it is still efficient to serve $\hat{q}$ agents; in the second case, no winner can be replaced.

Finally, we use the structure of the exclusion pivot transfer policy to prove that whether or not winners are replaceable, and no matter how many winners between $\check{q}$ and $\hat{q}$ are selected, we have that (i) no loser envies a loser, (ii) no winner envies a loser, (iii) no winner envies a winner, and (iv) no loser envies a winner.

Lemma B.3. *Fix an environment. If the cost function is* convex *and* symmetric, *then each exclusion pivot mechanism satisfies* no-envy.

Proof. Fix an environment whose cost function is *convex* and *symmetric*, and let (τ, φ) be an exclusion pivot mechanism. Since the cost function is symmetric, we abuse notation and write $C : \{0, 1, 2, \dots, n\} \to \mathbb{R} \cup \{\infty\}$ so that $C(q)$ denotes the cost of producing any q winners, and write $\mathcal{Q} \subseteq \{0, 1, \dots, n\}$ to denote the quantities of winners that can be produced for finite cost. By Lemma A.1, (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *voluntary participation*. We want to show that (τ, φ) satisfies *no-envy*; thus let $v \in V$. If $n = 1$ then we are done, so assume $n \geq 2$.

To avoid confusion, we refer to agents as members of N and nonzero quantities as members of $\{1, 2, \dots, n\}$, even though these are the same set. We also introduce a ranking of the agents to be used with the nonzero quantity notation: (i) label the agents $A_1, A_2, \dots, A_n$ such that $v_{A_1} \geq v_{A_2} \geq \cdots \geq v_{A_n}$, and (ii) for each $i \in N$, define $q_i \in \{1, 2, \dots, n\}$ to be the solution to $i = A_{q_i}$. Depending on the argument, we will sometimes refer to an agent as i and other times as A_q, but never as q (which is reserved for quantities).

For each $q \in \{1, 2, \dots, n\}$, define

$$\mathsf{S}_q \equiv \begin{cases} C(q) - C(q-1), & q \in \mathcal{Q}, \\ \infty, & \text{else}, \end{cases}$$

$$\mathsf{D}_q \equiv v_{A_q}.$$

We refer to $(\mathsf{S}_q)_{q \in \{1,2,\dots,n\}}$ as the *supply curve* and $(\mathsf{D}_q)_{q \in \{1,2,\dots,n\}}$ as the *demand curve*. Notice that S_q is the marginal cost of including a qth winner, so by *convexity*, we have $\mathsf{S}_1 \leq \mathsf{S}_2 \leq \cdots \leq \mathsf{S}_n$. By construction, $\mathsf{D}_1 \geq \mathsf{D}_2 \geq \cdots \geq \mathsf{D}_n$.

For each $i \in N$ and each $q \in \{1, 2, \dots, n-1\}$, define

$$\mathsf{D}_{q|i} \equiv \begin{cases} \mathsf{D}_q, & q < q_i, \\ \mathsf{D}_{q+1}, & q \geq q_i, \end{cases}$$

We refer to $(\mathsf{D}_{q|i})_{q\in\{1,2,\ldots,n-1\}}$ as the *demand curve given i is removed*. By construction, $\mathsf{D}_{1|i} \geq \mathsf{D}_{2|i} \geq \cdots \geq \mathsf{D}_{n-1|i}$.

Finally, define

$$\check{q} \equiv \max\bigl(\bigl\{q \in \{1, 2, \ldots, n\} | \mathsf{D}_q > \mathsf{S}_q\bigr\} \cup \{0\}\bigr),$$
$$\hat{q} \equiv \max\bigl(\bigl\{q \in \{1, 2, \ldots, n\} | \mathsf{D}_q \geq \mathsf{S}_q\bigr\} \cup \{0\}\bigr), \quad \text{and}$$
$$\text{for each } i \in N, \quad \hat{q}_i \equiv \max\bigl(\bigl\{q \in \{1, 2, \ldots, n-1\} | \mathsf{D}_{q|i} \geq \mathsf{S}_q\bigr\} \cup \{0\}\bigr).$$

We refer to $\check{q}$ as the *minimum efficient quantity*, $\hat{q}$ as the *maximum efficient quantity*, and $\hat{q}_i$ as the *maximum efficient quantity given i is removed*; these terms will be justified by the proof.

∘ Step 1: We have $|\varphi(v)| \in \{\check{q}, \check{q}+1, \ldots, \hat{q}\}$ and $[\max_{W\subseteq N} \sigma_v(W)] = \sum_{q=1}^{\hat{q}} (\mathsf{D}_q - \mathsf{S}_q)$.

For each $q \in \{0, 1, \ldots, n\}$, define $\sigma^*(q) \equiv \sigma_v(\{A_{q'} \in N | q' \leq q\})$ to be the surplus of serving the q highest-ranked agents at v. Then for each $q \in \{0, 1, \ldots, n\}$, we have

$$\sigma^*(q) = \begin{cases} \displaystyle\sum_{q'=1}^{q} (\mathsf{D}_{q'} - \mathsf{S}_{q'}), & q \in \mathcal{Q}, \\ -\infty, & \text{else.} \end{cases}$$

Since D_q is nonincreasing in q and S_q is nondecreasing in q, thus over $\mathcal{Q}$ we have that $(\mathsf{D}_q - \mathsf{S}_q)$ is nonincreasing in q. Then σ^* is maximized with any finite sum that includes all terms $(\mathsf{D}_q - \mathsf{S}_q)$ that are positive and none that are negative, so its maximizers are $\{\check{q}, \check{q}+1, \ldots, \hat{q}\}$.

For each $W \subseteq N$ such that $|W| \notin \{\check{q}, \check{q}+1, \ldots, \hat{q}\}$, we have $\sigma_v(\{A_1, A_2, \ldots, A_{\hat{q}}\}) = \sigma^*(\hat{q}) > \sigma^*(|W|) \geq \sigma_v(W)$, so by *production efficiency*, $\varphi(v) \neq W$; thus $|\varphi(v)| \in \{\check{q}, \check{q}+1, \ldots, \hat{q}\}$, as desired. For each $W \subseteq N$, we have $\sigma_v(\{A_1, A_2, \ldots, A_{\hat{q}}\}) = \sigma^*(\hat{q}) \geq \sigma^*(|W|) \geq \sigma_v(W)$; thus $[\max_{W\subseteq N} \sigma_v(W)] = \sigma^*(\hat{q}) = \sum_{q=1}^{\hat{q}} (\mathsf{D}_q - \mathsf{S}_q)$, as desired.

∘ Step 2: For each $i \in N$, $[\max_{W\subseteq N\setminus\{i\}} \sigma_{(0,v_{-i})}(W)] = \sum_{q=1}^{\hat{q}_i} (\mathsf{D}_{q|i} - \mathsf{S}_q)$.

The argument is analogous to that in the previous step.

∘ Step 3: For each $i \in \varphi(v)$, we have (i) for each $j \in N\setminus\varphi(v)$, $v_i \geq v_j$; and (ii) $v_i \geq \mathsf{D}_{\hat{q}}$.

Let $i \in \varphi(v)$. For each $j \in N\setminus\varphi(v)$, we must have $v_i \geq v_j$; else $\sigma_v((\varphi(v)\setminus\{i\}) \cup \{j\}) > \sigma_v(\varphi(v))$, contradicting *production efficiency*. Thus v_i is among the $|\varphi(v)|$ highest valuations, so by Step 1, v_i is among the $\hat{q}$ highest valuations; thus $v_i \geq \mathsf{D}_{\hat{q}}$.

∘ Step 4: If $\hat{q} = n$, then for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$.

Assume $\hat{q} = n$ and let $i \in \varphi(v)$. Since $\mathsf{D}_{n-1} \geq \mathsf{D}_n \geq \mathsf{S}_n \geq \mathsf{S}_{n-1}$, it follows immediately that $\mathsf{D}_{n-1|i} \geq \mathsf{S}_{n-1}$, so $\hat{q}_i = n - 1 = \hat{q} - 1$, as desired.

∘ Step 5: If $\hat{q} \in \{1, 2, \ldots, n-1\}$, then (i) $\mathsf{D}_{\hat{q}+1} \geq \mathsf{S}_{\hat{q}}$ implies for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$; and (ii) $\mathsf{D}_{\hat{q}+1} < \mathsf{S}_{\hat{q}}$ implies for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$.

Assume $\hat{q} \in \{1, 2, \ldots, n-1\}$ and let $i \in \varphi(v)$. Recall that $i = A_{q_i}$, while $\hat{q}_i$ is the maximum efficient quantity given i is removed. We consider three cases.

Case 1: If $\hat{q} < q_i$, then

- for each $q \in \{1, 2, \ldots, \hat{q}\}$, $\mathsf{D}_{q|i} = \mathsf{D}_q \geq \mathsf{S}_q$,
- for each $q \in \{\hat{q}+1, \hat{q}+2, \ldots, q_i - 1\}$, $\mathsf{D}_{q|i} = \mathsf{D}_q < \mathsf{S}_q$, and
- for each $q \in \{q_i, q_i + 1, \ldots, n-1\}$, $\mathsf{D}_{q|i} = \mathsf{D}_{q+1} \leq \mathsf{D}_q < \mathsf{S}_q$,

so $\hat{q}_i = \hat{q}$.

Case 2: If $q_i \leq \hat{q}$ and $\mathsf{D}_{\hat{q}+1} \geq \mathsf{S}_{\hat{q}}$, then

- for each $q \in \{1, 2, \ldots, q_i - 1\}$, $\mathsf{D}_{q|i} = \mathsf{D}_q \geq \mathsf{S}_q$,
- for each $q \in \{q_i, q_i + 1, \ldots, \hat{q} - 1\}$, $\mathsf{D}_{q|i} = \mathsf{D}_{q+1} \geq \mathsf{S}_{q+1} \geq \mathsf{S}_q$,
- $\mathsf{D}_{\hat{q}|i} = \mathsf{D}_{\hat{q}+1} \geq \mathsf{S}_{\hat{q}}$, and
- for each $q \in \{\hat{q}+1, \hat{q}+2, \ldots, n-1\}$, $\mathsf{D}_{q|i} = \mathsf{D}_{q+1} \leq \mathsf{D}_q < \mathsf{S}_q$,

so $\hat{q}_i = \hat{q}$.

Case 3: If $q_i \leq \hat{q}$ and $\mathsf{D}_{\hat{q}+1} < \mathsf{S}_{\hat{q}}$, then we have the same bullets as in Case 2 except that the third is changed to $\mathsf{D}_{\hat{q}|i} = \mathsf{D}_{\hat{q}+1} < \mathsf{S}_{\hat{q}}$, so $\hat{q}_i = \hat{q} - 1$.

If $\mathsf{D}_{\hat{q}+1} \geq \mathsf{S}_{\hat{q}}$, then we are in Case 1 or Case 2, so $\hat{q}_i = \hat{q}$. If $\mathsf{D}_{\hat{q}+1} < \mathsf{S}_{\hat{q}}$, then since $i \in \varphi(v)$, thus by Step 3 we have $v_i \geq \mathsf{D}_{\hat{q}} \geq \mathsf{S}_{\hat{q}} > \mathsf{D}_{\hat{q}+1}$; then $q_i \leq \hat{q}$, so we are in Case 3 and $\hat{q}_i = \hat{q} - 1$. Since $i \in \varphi(v)$ was arbitrary, we are done.

∘ Step 6: For each $i \in \varphi(v)$, $\sum_{q=1}^{\hat{q}_i} \mathsf{D}_{q|i} = \sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q - v_i$.

Let $i \in \varphi(v)$, and recall $v_i = \mathsf{D}_{q_i}$. If $q_i \leq \hat{q}_i$, then

$$\begin{aligned}
\sum_{q=1}^{\hat{q}_i} \mathsf{D}_{q|i} &= \sum_{q=1}^{q_i-1} \mathsf{D}_{q|i} + \sum_{q=q_i}^{\hat{q}_i} \mathsf{D}_{q|i} + [\mathsf{D}_{q_i} - v_i] \\
&= \sum_{q=1}^{q_i-1} \mathsf{D}_q + \sum_{q=q_i}^{\hat{q}_i} \mathsf{D}_{q+1} + [\mathsf{D}_{q_i} - v_i] \\
&= \sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q - v_i,
\end{aligned}$$

as desired.

If $q_i > \hat{q}_i$, then $q_i \geq \hat{q}_i + 1$, so $v_i = \mathsf{D}_{q_i} \leq \mathsf{D}_{\hat{q}_i+1}$. Moreover, since $i \in \varphi(v)$, thus by Step 1, $\hat{q} \neq 0$; so by Step 4 and Step 5, $\hat{q} \leq \hat{q}_i + 1$; so by Step 3, $v_i \geq \mathsf{D}_{\hat{q}} \geq \mathsf{D}_{\hat{q}_i+1}$. Then $v_i = \mathsf{D}_{\hat{q}_i+1}$, so

$$\begin{aligned}\sum_{q=1}^{\hat{q}_i} \mathsf{D}_{q|i} &= \sum_{q=1}^{\hat{q}_i} \mathsf{D}_q + [\mathsf{D}_{\hat{q}_i+1} - v_i] \\ &= \sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q - v_i,\end{aligned}$$

as desired.

∘ Step 7: Conclude.

Since (τ, φ) is an exclusion pivot mechanism, thus for each $i \in N$, we have

$$\tau_i(v) = \left(\left[\max_{W \subseteq N} \sigma_v(W)\right] - \varphi_i(v) v_i\right) - \left[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)\right].$$

For each $i \notin \varphi(v)$, clearly $\varphi(v)$ maximizes $\sigma_{(0, v_{-i})}$ among subsets of $N \setminus \{i\}$, so

$$\begin{aligned}\tau_i(v) &= \sigma_v(\varphi(v)) - \sigma_{(0, v_{-i})}(\varphi(v)) \\ &= 0.\end{aligned}$$

Thus no loser envies another loser. Moreover, since (τ, φ) satisfies *voluntary participation*, no winner envies a loser. If $\hat{q} = 0$, then by Step 1 everybody is a loser and we are done; thus let us assume $\hat{q} \in \{1, 2, \dots, n\}$. By Step 4 and Step 5, either (i) for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$; or (ii) for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$. We consider both cases.

Case 1: For each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$. Then by Step 1, Step 2, and Step 6, for each $i \in \varphi(v)$, we have

$$\begin{aligned}\tau_i(v) &= \left(\left[\sum_{q=1}^{\hat{q}} \mathsf{D}_q\right] - C(\hat{q}) - v_i\right) - \left(\left[\sum_{q=1}^{\hat{q}_i} \mathsf{D}_{q|i}\right] - C(\hat{q}_i)\right) \\ &= \left(\left[\sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q\right] - C(\hat{q}_i + 1) - v_i\right) - \left(\left[\sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q\right] - v_i - C(\hat{q}_i)\right) \\ &= -\mathsf{S}_{\hat{q}_i+1} \\ &= -\mathsf{S}_{\hat{q}}.\end{aligned}$$

In this case, no winner envies another winner.

CASE 2: For each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$. Then by Step 1, Step 2, and Step 6, for each $i \in \varphi(v)$, we have

$$\begin{aligned}\tau_i(v) &= \left(\left[\sum_{q=1}^{\hat{q}} \mathsf{D}_q\right] - C(\hat{q}) - v_i\right) - \left(\left[\sum_{q=1}^{\hat{q}_i} \mathsf{D}_{q|i}\right] - C(\hat{q}_i)\right) \\ &= \left(\left[\sum_{q=1}^{\hat{q}_i} \mathsf{D}_q\right] - C(\hat{q}_i) - v_i\right) - \left(\left[\sum_{q=1}^{\hat{q}_i+1} \mathsf{D}_q\right] - v_i - C(\hat{q}_i)\right) \\ &= -\mathsf{D}_{\hat{q}_i+1} \\ &= -\mathsf{D}_{\hat{q}+1}.\end{aligned}$$

In this case, no winner envies another winner.

Assume, by way of contradiction, that a loser i envies a winner. If $\hat{q} = n$, then by Step 4 we are in Case 1; so $v_i > \mathsf{S}_n$, so $\sigma_v(|\varphi(v)| \cup \{i\}) > \sigma_v(\varphi(v))$, contradicting *production efficiency*. If $\hat{q} \in \{1, 2, \dots, n-1\}$, then by Step 5 and the above cases, either $\mathsf{D}_{\hat{q}+1} \geq \mathsf{S}_{\hat{q}}$ and $v_i > \mathsf{D}_{\hat{q}+1}$ or $\mathsf{D}_{\hat{q}+1} < \mathsf{S}_{\hat{q}}$ and $v_i > \mathsf{S}_{\hat{q}}$. In both cases, we have $v_i > \mathsf{D}_{\hat{q}+1}$ and $v_i > \mathsf{S}_{\hat{q}}$. Define

$$q^* \equiv \max\bigl\{q \in \{1, 2, \dots, n\} \big| \mathsf{D}_{q^*} = v_i\bigr\}.$$

Since $\mathsf{D}_{q^*} = v_i > \mathsf{D}_{\hat{q}+1}$, thus $q^* \leq \hat{q}$, so $\mathsf{D}_{q^*} = v_i > \mathsf{S}_{\hat{q}} \geq \mathsf{S}_{q^*}$. Then $q^* \leq \check{q}$, so by Step 1 there must be at least q^* winners, so by Step 3 i must be a winner, contradicting that i is a loser. □

To conclude, Theorem 2 is a direct corollary of our previous lemmas.

THEOREM 2 (Repeated). *Fix an environment. The following are equivalent:*

- *there is a mechanism that satisfies* production efficiency, strategy-proofness, *and* no-envy*;*
- *each exclusion pivot mechanism satisfies* production efficiency, strategy-proofness, *and* no-envy*; and*
- *the cost function satisfies* convexity *and* symmetry.

PROOF. The first item implies the third by Lemma B.1 and Lemma B.2, the third implies the second by Lemma B.3, and the second implies the first trivially. □

APPENDIX C

In this appendix, we establish seven propositions providing logical relationships between *inclusion cost coverage* and other standard properties of cost functions. In addition to the properties introduced in Section 3.1, we consider the following.

Definition. Fix an environment. The cost function satisfies

- *monotonicity* if and only if for each pair $W, W' \subseteq N$ such that $W \subseteq W'$, we have $C(W') \geq C(W)$; and
- *superadditivity* if and only if for each pair $W, W' \subseteq N$ such that $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$, we have (i) $W, W' \in \mathcal{W}$, and (ii) $C(W \cup W') \geq C(W) + C(W')$.

Finally, if the cost function is *symmetric*, then for each $w \in \{0, 1, \ldots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. In this case, the cost function satisfies

- *nondecreasing average costs* if and only if for each pair $w, w' \in \{1, 2, \ldots, n\}$ such that $w' > w$,
  - (i) $C(w) = \infty$ implies $C(w') = \infty$, and
  - (ii) if $C(w), C(w') \in \mathbb{R}$, then $\frac{C(w)}{w} \leq \frac{C(w')}{w'}$.

In particular, we establish that (i) *convexity* implies *inclusion cost coverage* (Proposition 1); (ii) *inclusion cost coverage* does not imply *monotonicity* (Proposition 2); (iii) under *symmetry*, *inclusion cost coverage* is equivalent to *nondecreasing average costs* (Proposition 3); (iv) *symmetry* and *inclusion cost coverage* do not together imply *convexity* (Proposition 4); (v) *symmetry* and *inclusion cost coverage* together imply *superadditivity* (Proposition 5); (vi) *symmetry* and *superadditivity* do not together imply *inclusion cost coverage* (Proposition 6); and (vii) *monotonicity* and *inclusion cost coverage* do not together imply *superadditivity* (Proposition 7).

Proposition 1. *Fix a set of agents. If a cost function satisfies* convexity*, then it satisfies* inclusion cost coverage*.*

Proof. Let C be *convex* and let $W \in \mathcal{W}$ be nonempty. Define $w \equiv |W|$, and re-index the agents such that $W = \{1, 2, \ldots, w\}$. Define $W_0 \equiv \emptyset$, and for each $i \in W$, define $W_i \equiv \{1, 2, \ldots, i\}$. For each $i \in W$, by *convexity* we have $\mathcal{I}_i(W) \geq \mathcal{I}_i(W_i)$; thus as $W \in \mathcal{W}$, we must have $W_i \in \mathcal{W}$.

For each $i \in W$, since $W_i \in \mathcal{W}$ and $W_{i-1} \in \mathcal{W}$, thus $\mathcal{I}_i(W_i) \in \mathbb{R}$, so by *convexity* we have $\mathcal{I}_i(W) \neq -\infty$, so as $W \in \mathcal{W}$ we have $\mathcal{I}_i(W) \in \mathbb{R}$. Thus by *convexity*,

$$
\begin{aligned}
C(W) &= \big[C(W_w) - C(W_{w-1})\big] + \big[C(W_{w-1}) - C(W_{w-2})\big] + \cdots + \big[C(W_1) - C(W_0)\big] \\
&= \mathcal{I}_w(W_w) + \cdots + \mathcal{I}_1(W_1) \\
&\leq \mathcal{I}_w(W) + \cdots + \mathcal{I}_1(W) \\
&= \sum_{i \in W} \mathcal{I}_i(W).
\end{aligned}
$$

Altogether, since W was an arbitrary nonempty member of $\mathcal{W}$, thus C satisfies *inclusion cost coverage*, as desired. $\square$

Proposition 2. *A cost function may satisfy* inclusion cost coverage *but not* monotonicity.

Proof. The proof is by example. Let $N = \{1, 2, 3\}$, and define C as follows:

$$C(W) = \begin{cases} 1, & W = N, \\ 2, & W = \{1, 2\}, \\ 0, & \text{else.} \end{cases}$$

It is straightforward to verify C that satisfies *inclusion cost coverage* but not *monotonicity*. □

Proposition 3. *Fix a set of agents. If a cost function satisfies* symmetry*, then it satisfies* inclusion cost coverage *if and only if it satisfies* nondecreasing average costs.

Proof. Let C satisfy *symmetry*. For each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. We establish both implications:

[$\Rightarrow$] Assume C satisfies *inclusion cost coverage*, and let $w, w' \in \{1, 2, \dots, n\}$ such that $w' > w$. Let $W, W' \in 2^N$ such that $|W| = w$ and $|W'| = w'$, and reindex the agents such that $W' \backslash W = \{1, 2, \dots, w' - w\}$. Define $W_0 \equiv W$, and for each $i \in W' \backslash W$, define $W_i \equiv W_{i-1} \cup \{i\}$.

If $C(w) = \infty$, then $W = W_0 \notin \mathcal{W}$. By *inclusion cost coverage*, for each $i \in W' \backslash W$, $W_{i-1} \notin \mathcal{W}$ implies $W_i \notin \mathcal{W}$; thus $W' = W_{w'-w} \notin \mathcal{W}$, so $C(w') = \infty$.

If $C(w), C(w') \in \mathbb{R}$, then $W' = W_{w'-w} \in \mathcal{W}$. By *inclusion cost coverage*, for each $i \in W' \backslash W$, $W_i \in \mathcal{W}$ implies $W_{i-1} \in \mathcal{W}$; thus by *inclusion cost coverage* and *symmetry*, for each $i \in W' \backslash W$, we have

$$\begin{aligned} C(w + i) &= C(W_i) \\ &\leq |W_i|\big[C(W_i) - C(W_{i-1})\big] \\ &= (w + i)C(w + i) - (w + i)C(w + i - 1), \end{aligned}$$

or $\frac{C(w+i-1)}{w+i-1} \leq \frac{C(w+i)}{w+i}$. Altogether, then, $\frac{C(w)}{w} \leq \frac{C(w')}{w'}$.

Since $w, w' \in \{1, 2, \dots, n\}$ such that $w' > w$ were arbitrary, thus C satisfies *nondecreasing average costs*, as desired.

[$\Leftarrow$] Assume C satisfies *nondecreasing average costs*. Let $W \in \mathcal{W}$ be nonempty and define $w \equiv |W|$. Then $C(w) \neq \infty$. By *nondecreasing average costs*, $C(w - 1) \neq \infty$, so for each $i \in W$, $W \backslash \{i\} \in \mathcal{W}$. Moreover, by *nondecreasing average costs*, $\frac{C(w-1)}{w-1} \leq \frac{C(w)}{w}$, so $wC(w - 1) \leq wC(w) - C(w)$, so $C(w) - C(w - 1) \geq \frac{C(w)}{w}$. Thus for each $i \in W$, we have $\mathcal{I}_i(W) \geq \frac{C(W)}{|W|}$, so $\sum_W \mathcal{I}_i(W) \geq C(W)$. Since W was an arbitrary nonempty member of $\mathcal{W}$, thus C satisfies *inclusion cost coverage*, as desired. □

Proposition 4. *A cost function may satisfy* symmetry *and* inclusion cost coverage *but not* convexity.

Proof. This is established by the cost function used as an example in the proof sketch of Theorem 2. □

Proposition 5. *Fix a set of agents. If a cost function satisfies* symmetry *and* inclusion cost coverage, *then it satisfies* superadditivity.

Proof. Let C satisfy *symmetry* and *inclusion cost coverage*. For each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. By Proposition 3, C satisfies *nondecreasing average costs*.

Let $W, W' \subseteq N$ such that $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$. Define $w \equiv |W|$ and $w' \equiv |W'|$. Then $C(w+w') \neq \infty$, so by *nondecreasing average costs*, we have $C(w) \neq \infty$ and $C(w') \neq \infty$, so $W \in \mathcal{W}$ and $W' \in \mathcal{W}$. If $w = 0$ or $w' = 0$, then $C(W \cup W') = C(w+w') = C(w) + C(w') = C(W) + C(W')$, as desired. If $w, w' \in \{1, 2, \dots, n\}$, then by *nondecreasing average costs*,

$$\begin{aligned} C(W) + C(W') &= C(w) + C(w') \\ &= w\frac{C(w)}{w} + w'\frac{C(w')}{w'} \\ &\leq w\frac{C(w+w')}{w+w'} + w'\frac{C(w+w')}{w+w'} \\ &= C(w+w') \\ &= C(W \cup W'), \end{aligned}$$

as desired. Since $W, W' \subseteq N$ with $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$ were arbitrary, thus C satisfies *superadditivity*, as desired. □

Proposition 6. *Fix a set of agents. A cost function may satisfy* symmetry *and* superadditivity *but not* inclusion cost coverage.

Proof. The proof is by example. Let $N = \{1, 2, 3\}$, and define C as follows:

$$C(W) = \begin{cases} 0, & |W| = 0, \\ 1, & |W| = 1, \\ 4, & |W| = 2, \\ 5, & |W| = 3. \end{cases}$$

It is straightforward to verify that C satisfies *symmetry* and *superadditivity* but not *inclusion cost coverage*. □

Proposition 7. *Fix a set of agents. A cost function may satisfy* monotonicity *and* inclusion cost coverage, *but not* superadditivity.

Proof. The proof is by example. Let $N = \{1, 2, 3, 4\}$, and define C as follows:

$$C(W) = \begin{cases} 4, & W = N, \\ 3, & |W| = 3, W = \{1, 2\}, \text{ or } W = \{3, 4\}, \\ 0, & \text{else.} \end{cases}$$

It is straightforward to verify that C satisfies *monotonicity* and *inclusion cost coverage*, but $C(\{1, 2, 3, 4\}) < C(\{1, 2\}) + C(\{3, 4\})$, and thus C does not satisfy *superadditivity*. □

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