Negotiation across Multiple Issues
Online Appendix

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This appendix contains the non-emptiness characterizations of the sum of the cores of the individual issues ($\sum_{V_j \in \bar{V}} C(V_j)$) and of the core of the sum of individual issues ($C(\sum_{V_j \in \bar{V}} V_j)$). These characterizations use systems of multi-weights, which makes them comparable to the non-emptiness characterization of the multi-core (Theorem 2 in the paper). For this purpose two additional sets of systems of multi-weights are presented together with the systems of multi-weights that appear in Definition 6 in the paper.

1 Definitions

1.1 Multi-weights

A function $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ that assigns a non-negative real number to every triplet of coalition, agent, and issue is a system of multi-weights.

We concentrate on systems of multi-weights that satisfy Zero to Non-members and Resource Exhaustion.

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Definition 1. *A system of multi-weights, $\tilde{\delta}$, satisfies Zero to Non-members if* $\forall V_j \in \bar{V}, \forall i \in N, \forall S \in 2^{N \setminus \{i\}} : \tilde{\delta}(S, i, V_j) = 0.$

Zero to Non-members entails a system of multi-weights that assigns zero weight to all triplets where the agent is not a member of the coalition.

Definition 2. *A system of multi-weights, $\tilde{\delta}$, satisfies Resource Exhaustion if* $\forall V_j \in \bar{V} : \sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \chi^N.$

Resource Exhaustion implies that each agent is endowed with one unit of time per issue. When Resource Exhaustion and Zero to Non-members are imposed, we refer to a system of multi-weights as an (unrestricted) system of balancing multi-weights.[1]

The following two definitions impose across-issue restrictions on systems of multi-weights. Definition 3 requires that the total weights (over coalitions) assigned to triplets that include Agent i be constant across issues. Definition 4 compels the weights assigned to triplets that include Agent i and Coalition S to be the same across issues.

Definition 3. *A system of multi-weights, $\tilde{\delta}$, satisfies Constant Shares if* $\forall i \in N, \forall V_j, V_{j'} \in \bar{V} : \sum\limits_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum\limits_{S \in 2^N} \tilde{\delta}(S, i, V_{j'})\chi^S.$

Definition 4. *A system of multi-weights, $\tilde{\delta}$, satisfies Constant Allocations if* $\forall i \in N, \forall V_j, V_{j'} \in \bar{V}, \forall S \in 2^N : \tilde{\delta}(S, i, V_j) = \tilde{\delta}(S, i, V_{j'}).$

[1] To see that balancedness is imposed in each Issue V_j, set $\delta(S) = \sum_{i \in N} \tilde{\delta}(S, i, V_j)$. Then, Resource Exhaustion implies that in each issue V_j, $\sum_{S \in 2^N} \delta(S)\chi^S = \chi^N$. Observe that the identity of Agent i is ignored in $\delta(S)$, therefore, when restricting attention to Issue V_j, several systems of balancing multi-weights are reduced to one system of balancing weights. Conversely, every system of balancing weights corresponds to at least one system of balancing multi-weights (e.g. dividing $\delta(S)$ equally among the members of S).

1.2 Systems

We concentrate on the following three families of systems of balancing multi-weights:

Definition 5. *A function* $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ *that satisfies Zero to Non-members and Resource Exhaustion is*

1. *a system of Unconstrained Balancing Multi-weights* (Δ_{UC} *is the set of all systems of unconstrained balancing multi-weights).*

2. *a system of Balancing Multi-weights if it satisfies Constant Shares* (Δ *is the set of all systems of balancing multi-weights).*

3. *a system of Balancing Multi-weights with Constant Allocations if it satisfies Constant Allocations* (Δ_{CA} *is the set of all systems of balancing multi-weights with constant allocations).*

The Constant Allocations requirement implies the Constant Shares requirement, but not the opposite. Therefore, $\Delta_{UC} \supseteq \Delta \supseteq \Delta_{CA}$. The difference between the three definitions lies in the dependencies they impose on the weights across issues. The elements of Δ_{UC} are unrestricted across issues, so that $\tilde{\delta}(\cdot, \cdot, V_j)$ poses no restriction on the values of $\tilde{\delta}(\cdot, \cdot, V_{j'})$, for every $V_j, V_{j'} \in \bar{V}$. By contrast, for Δ_{CA}, $\tilde{\delta}(\cdot, \cdot, V_j)$ and $\tilde{\delta}(\cdot, \cdot, V_{j'})$ must be the same for every $V_j, V_{j'} \in \bar{V}$. The set Δ, that lies between these two sets, allows for some variation of $\tilde{\delta}(\cdot, \cdot, V_j)$ across issues, so long as they obey the Constant Shares requirement.[2]

[2]Put differently, consider the set of functions that assign weights to agent-coalition pairs restricted by two requirements– assigning zero to pairs where the agent is not an element of the coalition and allocating a total weight of one to each agent across coalitions,

$$F = \left\{ f : N \times 2^N \to \mathbb{R}_+ \middle| i \notin S \text{ implies } f(i,S) = 0, \forall i \in N : \sum_{S \in \left\{ T \cup \{i\} | T \subseteq N \setminus \{i\} \right\}} \sum_{k \in S} f(k,S) = 1 \right\}$$

The three sets, Δ_{UC}, Δ and Δ_{CA}, coincide when the multi-issue problem consists of only one issue V. The correspondence above between standard weights and multi-weights, establishes that any collection of coalitions that are assigned positive weights in some system of balancing weights can also be assigned positive weights by any one of the three definitions above.

This observation is still true when concentrating on the weights of a specific issue in the multi-game. However, once these weights are set, definitions 5.2 and 5.3 confine the possible weights in the other issues.

2 Example

The table below presents three examples of systems of balancing multi-weights with $\tilde{\delta}_1$, $\tilde{\delta}_2$ and $\tilde{\delta}_3$, corresponding to the three definitions above in a two-issue three-agent multi-game. A row in this table corresponds to a triplet – coalition, agent, and issue.[3] The Constant Allocation condition is satisfied by $\tilde{\delta}_3$ since for every Agent i and for every Coalition S, $\tilde{\delta}_3(S,i,V_1) = \tilde{\delta}_3(S,i,V_2)$, whereas the two other functions do not satisfy it (e.g. Agent 1 and Coalition $\{1,2\}$). The Constant Shares condition is satisfied by both $\tilde{\delta}_2$ and $\tilde{\delta}_3$, but is violated by $\tilde{\delta}_1$ (Agent 1).

For a given $V_j \in \bar{V}$, $\tilde{\delta}(S,i,V_j)$ satisfies Zero to Non-members and Resource Exhaustion if and only if it is an element of F.

Definition 5.1 states that Δ_{UC} is the set of systems of multi-weights where for each issue V_j, $\tilde{\delta}(S,i,V_j)$ is some element of F.

Definition 5.3 states that Δ_{CA} is the set of systems of multi-weights where for each issue V_j, $\tilde{\delta}(S,i,V_j)$ is the same element of F.

Let Π be a partition of F such that two functions f and f' belong to the same class if for every pair of agents i and k,

$$\sum_{S\in\left\{T\cup\{i,k\}|T\subseteq N\setminus\{i,k\}\right\}} f(i,S) = \sum_{S\in\left\{T\cup\{i,k\}|T\subseteq N\setminus\{i,k\}\right\}} f'(i,S)$$

Definition 5.2 states that Δ is the set of systems of multi-weights where for each issue V_j, $\tilde{\delta}(S,i,V_j)$ belongs to the same class of Π.

[3]Every triplet that is not specified in the table is assigned zero weight. Notice that both Zero to Non-members and Resource Exhaustion are satisfied by all three systems.

Issue	Agent	Coalition	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$
Issue V_1	Agent 1	$\{1\}$	0	0	0
		$\{1,2\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2,3\}$	0	0	0
	Agent 2	$\{2\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2\}$	0	0	0
		$\{2,3\}$	0	0	0
		$\{1,2,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	Agent 3	$\{3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
		$\{1,3\}$	0	0	$\frac{1}{8}$
		$\{2,3\}$	0	0	$\frac{1}{8}$
		$\{1,2,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
Issue V_2	Agent 1	$\{1\}$	1	$\frac{1}{4}$	0
		$\{1,2\}$	0	0	$\frac{1}{4}$
		$\{1,3\}$	0	0	$\frac{1}{4}$
		$\{1,2,3\}$	0	$\frac{1}{4}$	0
	Agent 2	$\{2\}$	0	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2\}$	0	0	0
		$\{2,3\}$	$\frac{1}{2}$	0	0
		$\{1,2,3\}$	0	$\frac{1}{4}$	$\frac{1}{4}$
	Agent 3	$\{3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$
		$\{1,3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$
		$\{2,3\}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$
		$\{1,2,3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$

Table 1: Three systems of balancing multi-weights

3 Results

Proposition 1. *The sum of the cores of the individual issues of* $\bar{V}$, $\sum\limits_{V_j \in \bar{V}} C(V_j)$, *is non-empty if and only if every* $\tilde{\delta} \in \Delta_{UC}$ *satisfies*

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

Proposition 2. *The core of the sum of individual issues of* $\bar{V}$, $C(\sum\limits_{V_j \in \bar{V}} V_j)$, *is non-empty if and only if every* $\tilde{\delta} \in \Delta_{CA}$ *satisfies*

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

Both proofs rely directly on the Bondareva-Shapley Theorem (Theorem 1 in the paper). Theorem 2 in the paper and Proposition 1 show that if there is no solution in the multi-core, the sum of the cores of the individual issues is empty as well, since $\Delta \subseteq \Delta_{UC}$. Theorem 2 in the paper and Proposition 2 show that if the core of the sum of individual issues is empty, so is the multi-core, since $\Delta_{CA} \subseteq \Delta$. These results are also established by Theorem 4 in the paper. The advantage of Propositions 1 and 2 is that they help identify the systems of balancing multi-weights that violate the conditions above when either $\sum_{V_j \in \bar{V}} C(V_j) = \emptyset$ and $M(\bar{V}) \neq \emptyset$, or $M(\bar{V}) = \emptyset$ and $C(\sum_{V_j \in \bar{V}} V_j) \neq \emptyset$.

4 Proof - Proposition 1

Proof. First, suppose $\sum\limits_{V_j \in \bar{V}} C(V_j) \neq \emptyset$. For every system of unconstrained balancing multi-weights, $\tilde{\delta} \in \Delta_{UC}$, let us define $\delta_j(S) = \sum_{i=1}^{n} \tilde{\delta}(S, i, V_j)$. For every

Issue V_j, $\delta_j(S)$ is a system of balancing weights since by Resource Exhaustion, $\sum_{S \in 2^N} \delta_j(S)\chi^S = \chi^N$.

Suppose there exists $\tilde{\delta}(S, i, V_j)$, such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

Then, there exists $V_j \in \bar{V}$ such that

$$V_j(N) < \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

or,

$$V_j(N) < \sum_{S \in 2^N} \delta_j(S) V_j(S)$$

By the Bondareva-Shapley Theorem, $C(V_j) = \emptyset$ and therefore $\sum_{V_j \in \bar{V}} C(V_j) = \emptyset$. Thus, every $\tilde{\delta} \in \Delta_{UC}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

For the other direction, suppose that every $\tilde{\delta} \in \Delta_{UC}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

For every $V_j \in \bar{V}$ and for every system of balancing weights $\delta(S)$, define $\tilde{\delta}(S, i, V_l)$ as follows,

1. If $V_l \neq V_j$ and $S \neq N$ then for every $i \in N$, $\tilde{\delta}(S, i, V_l) = 0$.

2. If $V_l \neq V_j$ and $S = N$ then for every $i \in N$, $\tilde{\delta}(N, i, V_l) = \frac{1}{n}$.

3. If $V_l = V_j$ then $\tilde{\delta}(S, i, V_j) = \frac{\delta(S)}{|S|}$ if $i \in S$ and 0 otherwise.

Note that $\tilde{\delta}$ satisfies the Zero to Non-members condition. Also, for $V_l \neq V_j$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_l)\chi^S = \sum_{i \in N} \tilde{\delta}(N, i, V_l)\chi^N = \sum_{i \in N} \frac{1}{n}\chi^N = \chi^N$$

and for $V_l = V_j$

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum_{S \in 2^N} \sum_{i \in S} \frac{\delta(S)}{|S|}\chi^S = \sum_{S \in 2^N} \delta(S)\chi^S = \chi^N$$

Therefore, $\tilde{\delta}(S, i, V_l)$ also satisfies the Resources Exhaustion condition and therefore it is a system of unconstrained balancing multi-weights.

Suppose, there exists an issue $V_j \in \bar{V}$ such that $C(V_j) = \emptyset$. Then, by the Bondareva-Shapley Theorem, there exists a system of balancing weights, $\delta_j(S)$, such that $V_j(N) < \sum_{S \in 2^N} \delta_j(S)V_j(S)$. Consider the corresponding $\tilde{\delta}$,

$$\begin{aligned}
&\sum_{V_l \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_l)V_l(S) = \\
&\sum_{V_l \in \bar{V} \setminus \{V_j\}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_l)V_l(S) + \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)V_j(S) = \\
&\sum_{V_l \in \bar{V} \setminus \{V_j\}} \sum_{i=1}^{n} \frac{1}{n} V_l(N) + \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)V_j(S) = \\
&\sum_{V_l \in \bar{V} \setminus \{V_j\}} V_l(N) + \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)V_j(S) = \\
&\sum_{V_l \in \bar{V} \setminus \{V_j\}} V_l(N) + \sum_{S \in 2^N} \sum_{i \in S} \frac{\delta(S)}{|S|} V_j(S) = \\
&\sum_{V_l \in \bar{V} \setminus \{V_j\}} V_l(N) + \sum_{S \in 2^N} \delta(S)V_j(S) > \sum_{V_l \in \bar{V} \setminus \{V_j\}} V_l(N) + V_j(N) = \sum_{V_l \in \bar{V}} V_l(N)
\end{aligned}$$

Therefore, it must be that $\forall V_j \in \bar{V} : C(V_j) \neq \emptyset$ and therefore $\sum\limits_{V_j \in \bar{V}} C(V_j) \neq \emptyset$. $\square$

5 Proof - Proposition 2

Proof. Suppose $C(\sum\limits_{V_j \in \bar{V}} V_j) \neq \emptyset$. Assume by negation that there exists $\tilde{\delta} \in \Delta_{CA}$ such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{V_j \in V} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

or,

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \sum_{i=1}^{n} \sum_{V_j \in V} \tilde{\delta}(S, i, V_j) V_j(S)$$

Since $\tilde{\delta}$ is a system of balancing multi weights with constant allocation, for every agent i, coalition S and two issues V_j and $V_{j'}$:

$$\tilde{\delta}(S, i, V_j) = \tilde{\delta}(S, i, V_j') \equiv \tilde{\delta}(S, i)$$

and therefore,

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \sum_{i=1}^{n} \tilde{\delta}(S, i) \sum_{V_j \in \bar{V}} V_j(S)$$

Define $\delta(S) = \sum_{i=1}^{n} \tilde{\delta}(S, i)$. Due to the Resource Exhaustion property of $\tilde{\delta}$, $\delta(S)$ is a system of balancing weights

$$\sum_{S \in 2^N} \delta(S) \chi^S = \sum_{S \in 2^N} \left[\sum_{i=1}^{n} \tilde{\delta}(S, i) \right] \chi^S = \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i) \chi^S = \chi^N$$

Therefore, the inequality above becomes,

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S)$$

which by the Bondareva-Shapley Theorem implies that $C(\sum_{V_j \in \bar{V}} V_j) = \emptyset$, which is a contradiction. Thus, if $C(\sum_{V_j \in \bar{V}} V_j) \neq \emptyset$ then every $\tilde{\delta} \in \Delta_{CA}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

For the other direction, suppose $C(\sum_{V_j \in \bar{V}} V_j) = \emptyset$. Then, by the Bondareva-Shapley Theorem, there exists a system of balancing weights, $\delta(S)$, whereby $\sum_{S \in 2^N} \delta(S)\chi^S = \chi^N$ such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S)$$

Define $\tilde{\delta}(S, i, V_j) = \frac{\delta(S)}{|S|}$ if $i \in S$ and $\tilde{\delta}(S, i, V_j) = 0$ otherwise. Obviously, $\tilde{\delta}$ satisfies the Zero to Non-members condition. Also, for every $V_j \in \bar{V}$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum_{S \in 2^N} \sum_{i \in S} \frac{\delta(S)}{|S|}\chi^S = \sum_{S \in 2^N} \delta(S)\chi^S = \chi^N$$

Therefore, $\tilde{\delta}$ also satisfies the Resources Exhaustion condition. In addition, $\tilde{\delta}$ does not depend on any specific issue and thus it is a system of balancing multi-weights with constant allocations.

$$\sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S) = \sum_{S \in 2^N} \sum_{V_j \in \bar{V}} \sum_{i \in S} \frac{\delta(S)}{|S|} V_j(S)$$
$$= \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S) > \sum_{V_j \in \bar{V}} V_j(N)$$

Thus, if $C(\sum_{V_j \in \bar{V}} V_j) = \emptyset$ there exists $\tilde{\delta} \in \Delta_{CA}$ such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

$\square$