Ranking by Rating

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Abstract

Each item in a given collection is characterized by a set of possible performances. A (ranking) method is a function that assigns an ordering of the items to every performance profile. Ranking by Rating consists in evaluating each item’s performance by using an exogenous rating function, and ranking items according to their performance ratings. Any such method is separable: the ordering of two items does not depend on the performances of the remaining items. Consistency requires that if a change in an item’s performance improves its relative ranking against some other item at a given profile, the same change never decreases its relative ranking against any item at any profile. When performances belong to a finite set, ranking by rating is characterized by Separability and Consistency; this characterization generalizes to the infinite case under a continuity axiom. Consistency follows from Separability and Symmetry, or from Monotonicity alone. When performances are vectors in $\mathbb{R}_+^m$, a separable, symmetric, monotonic, continuous, and invariant method must rank items according to a weighted geometric mean of their performances along the $m$ dimensions.

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1 Introduction

Rankings are ubiquitous: we rank products and services such as cars, restaurants, scientific journals, webpages, songs; people such as athletes, students, chess players; institutions and groups such as schools, universities, academic departments, football teams, and even cities or countries.

Two types of ranking methods are widely used. Under the simplest and more traditional ones, an item’s performance is rated using a set of exogenously specified and weighted criteria, and the items are then ranked according to their ratings. This is, for instance, how a ranking of students

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is usually computed from their performances at an exam. Under more sophisticated methods, the weights of the criteria used to evaluate an item’s performance vary with the performances of all items. This is typically how academic journals are ranked according to the citations they receive from other journals, or webpages according to how they are linked to other pages: a reference from a highly ranked journal or webpage carries an endogenously greater weight.

Methods of the first type are separable: the ranking of two items does not depend on the performances of the remaining items. This offers a guarantee of transparency which probably accounts for the popularity of these methods. Methods of the second type are not separable.

In the current note, we are exclusively concerned with separable methods. Our formal model has \( n \) items, each of which is characterized by a possibly different set of conceivable performances. No a priori structure is imposed on this set. A ranking method is modelled as a function that computes an ordering of the items for every performance profile. We call ranking by rating the class of methods where each item’s performance is evaluated using an exogenous rating function defined over the set of its possible performances, and the items are ranked according to the resulting performance ratings. The main question we ask is whether all separable methods are of this type, and, if not, under which conditions that may be the case.

Our results are rather elementary and perhaps folk knowledge, but were, to the best of our knowledge, in need of a proof. We show in Section 3 that there exist separable methods other than ranking by rating. Those methods need not be degenerate and can be quite flexible; their range may include all the linear orderings of the items.

Theorem 1 in Section 4 shows that if the performance sets are finite, a ranking method is ranking by rating if and only if it is separable and consistent. Consistency here means that if a change in an item’s performance improves its relative ranking against some other item at a given profile, the same change does not decrease its relative ranking against any item at any profile. Corollary 1 extends Theorem 1 to the infinite case under an added continuity requirement.

Section 5 identifies two natural conditions under which a separable method is necessarily consistent. Symmetry, which applies when the performance sets of all items coincide, requires that permuting the performances of two items results in permuting their positions in the ranking. Monotonicity applies when each performance set is ordered, and requires that a higher performance improves an item’s position in the ranking. Theorem 2 shows that, in the finite case, a separable ranking method which is symmetric or monotonic is of the ranking-by-rating type, and Corollary 2 extends this conclusion to the infinite case under continuity.

Section 6 studies the particular case of our model where an items’ possible performances are multidimensional and partially ordered: they are represented by vectors in \( \mathbb{R}^m \). If the partial performances along the various dimensions are measured in non-comparable units, the ranking should not change when the partial performances of all items along a given dimension are multiplied by the same positive number. We show that a separable, symmetric, monotonic, and continuous method satisfying this invariance condition must rank items according to a weighted geometric
mean of their performances along the $m$ dimensions. Section 7 argues that, from an axiomatic viewpoint, the simple geometric mean method is a serious competitor of the non-separable methods proposed in the literature.

2 Related literature

A sizable literature addresses the problem of characterizing separable orderings defined over a set of multidimensional alternatives such as a subset of $\mathbb{R}_+^m$. Separability, in that literature, means that the ordering of two alternatives whose coordinates coincide along one dimension does not change with the value of that coordinate. The seminal contribution is that of Gorman (1968), who shows that, under suitable (and important) topological assumptions, such an ordering can be represented by an additively separable function. Bradley, Hodge and Kilgour (2005) show that Gorman’s result does not carry over to the finite case, and study properties of discrete separable orderings. Despite a formal similarity, our work is essentially unrelated to that literature. Even when the sets of possible performances of the $n$ items are infinite, we are interested in ordering only the finite sets containing precisely $n$ performances, one for each item. On the other hand, we want to order all such sets, and our separability condition is precisely a restriction on how these different rankings should be related: the ordering of two performances should not depend on what the remaining performances are.

Our separability condition is closely related to Arrow’s (1963) axiom of Independence of Irrelevant Alternatives and its weakening by Hansson (1973). Arrow’s aggregation problem, however, cannot be rephrased as a ranking problem of the type we analyze. To be sure, candidates (or social alternatives) may be regarded as items, and each candidate’s performance may be defined as the list of ranks he occupies in the preferences of the voters. But the set of possible performance profiles is not a Cartesian product: two candidates cannot both be ranked first by the same voter. Likewise, our separability condition is related to the independence condition used by Rubinstein (1980) to axiomatize the Copeland ranking method for tournaments, but the problem of ranking the participants in a tournament also lacks the Cartesian product structure imposed by our model.

A huge literature deals with the multidimensional sub-model discussed in Section 6 and, more specifically, with the case where items and dimensions coincide: this is indeed a suitable framework to discuss the popular issue of how to rank academic journals or webpages. In contrast with the current paper, that literature is concerned with non-separable methods and is therefore only tangentially related to our work.\footnote{Another difference is that, with a few exceptions such as Altman and Tennenholtz (2005), the literature focuses on cardinal methods. Such methods compute a score for each item, and the differences in scores are deemed meaningful.}

Two classes of methods have received considerable attention. The first consists of variants of the eigenvector solution based on the Perron-Frobenius theorem popularized by Landau (1895),
Wei (1952), Kendall (1955), Berge (1958), Keener (1993), and others: see Vigna (2009) for a survey. Under all such methods the ranking of the items is determined by their coordinates of the Perron vector of some irreducible nonnegative matrix, but the methods differ in how they construct this matrix from the performance data. For the problem of ranking webpages, examples include the PageRank method (Brin and Page, 1998), the HITS method (Kleinberg, 1999), and the SALSA method (Lempel and Moran, 2000): see Fercoq (2012) for a survey. For the problem of ranking journals, examples include the method of Leibowitz and Palmer (1984) and the intensity-invariant modification of Pinski and Narin (1976). Axiomatizations are offered by Palacios-Huerta and Volij (2004), Altman and Tennenholtz (2005), and Slutzki and Volij (2006).

The second class contains methods based on Sinkhorn’s (1967) algorithm for solving the so-called matrix scaling problem. For webpage ranking, examples of such methods include those of Smith (2005), Knight (2008), and Govan, Langville, and Meyer (2009). For the case where items and dimensions need not coincide, Demange (2014) proposes and axiomatizes the so-called handicap-based method.

3 Separability

Let \( N = \{1, \ldots, n\} \) be a finite set of items, \( n \geq 3 \). Each item \( i \in N \) is characterized by a nonempty set of possible performances \( A_i \). A performance profile is a list \( a = (a_1, \ldots, a_n) \in A_N := \times_{i \in N} A_i \). Let \( R_N \) denote the set of orderings\(^2\) on \( N \). A (ranking) method is a function \( R : A_N \to R_N \) that assigns to each performance profile \( a \) an ordering \( R(a) \) of the items. The statement \( (i, j) \in R(a) \), also written \( iR(a)j \), means that the method \( R \) considers \( i \) at least as strong as \( j \) when the performance profile is \( a \). Let \( P(a) \) and \( I(a) \) denote, respectively, the antisymmetric and symmetric components of \( R(a) \). If \( R(a) \) is a linear ordering, it will sometimes be convenient to express it by listing the items according to their rank: for instance, the linear ordering \( iR(a)j \iff i \leq j \) will be written \( R(a) = 1 \ 2 \ldots \ n \).

A method \( R \) is a ranking-by-rating method if there exist real-valued functions \( v_1, \ldots, v_n \) defined, respectively, on \( A_1, \ldots, A_n \), such that \( iR(a)j \iff v_i(a_i) \geq v_j(a_j) \) for all \( i, j \in N \) and all \( a \in A_N \). We call \( v_1, \ldots, v_n \) rating functions.

If \( R \) is a ranking-by-rating method, the relative ordering of two items depends only on the performances of these items. Formally, \( R \) satisfies the following property.

**Separability.** For all \( i, j \in N \) and \( a, a' \in A_N \), \[ a_i = a_i' \text{ and } a_j = a_j' \] \( \Rightarrow [iR(a)j \iff iR(a')j] \).

The following example shows that Separability does not characterize the ranking-by-rating methods.

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\(^2\)By an ordering we mean a complete, reflexive, and transitive binary relation. If this relation is also antisymmetric, we call it a linear ordering.
Example 1. Let $N = \{1, 2, 3\}$, $A_i = \{0, 1\}$ for all $i \in N$, and

$$R(a) = \begin{cases} 
1 2 3 & \text{if } a_1 = a_2, \\
2 1 3 & \text{if } a_1 \neq a_2.
\end{cases}$$

Since 3 is always ranked last, the relative ordering of 1 and 3 and the relative ordering of 2 and 3 are constant. The relative ordering of 1 and 2 varies, but it does not depend upon $a_3$. Thus, $R$ is separable. If the rating functions $v_1, v_2, v_3$ represented $R$, we should have

$$1P(0, 0, 0)2 \Rightarrow v_1(0) > v_2(0),$$
$$2P(1, 0, 0)1 \Rightarrow v_2(0) > v_1(1),$$
$$1P(1, 1, 0)2 \Rightarrow v_1(1) > v_2(1),$$
$$2P(0, 1, 0)1 \Rightarrow v_2(1) > v_1(0).$$

Since these inequalities are incompatible, $R$ is not a ranking-by-rating method.

In this example, the range of $R$ is very small. But there exist separable methods whose range contain all strict orderings on $N$ that are not ranking-by-rating methods. For instance, let $N = \{1, 2, 3\}$, $A_i = \{0, 1, 2\}$ for all $i \in N$, and consider the method $R$ depicted in Figure 1. It is tedious but straightforward to check that $R$ is separable, and the same argument as above shows that it is not a ranking-by-rating method.

4 Consistency

The separable method in Example 1 is inconsistent: a change in item 1’s performance from $a_1 = 0$ to $a_1 = 1$ improves that item’s position in the ranking when $(a_2, a_3) = (1, 1)$ but deteriorates it when $(a_2, a_3) = (0, 0)$. We show in this section that a separable method which does not exhibit this type of inconsistency is a ranking-by-rating method provided that (i) the set of performance profiles is finite or (ii) it is a connected topological space and the ranking method is continuous.

The following notation will be useful: if $i \in N$, $a \in A$, and $\alpha \in A_i$, then $(\alpha, a_{-i})$ is the performance profile obtained from $a$ by replacing $a_i$ with $\alpha$. We write $A_{-i}$ for $\times_{j \in N \setminus \{i\}} A_j$. The formal property relevant to our analysis is the following.

**Consistency.** For all $i \in N$ and $\alpha, \beta \in A_i$, if there exists $a_{-i} \in A_{-i}$ and $j \in N \setminus \{i\}$ such that $iP(\alpha, a_{-i})jR(\beta, a_{-i})i$ or $iR(\alpha, a_{-i})jP(\beta, a_{-i})i$, then there do not exist $b_{-i} \in A_{-i}$ and $k \in N \setminus \{i\}$ such that $iR(\beta, b_{-i})kR(\alpha, b_{-i})i$.

If, starting from some profile, the ranking of item $i$ relative to $j$ improves when $i$’s performance switches from $\beta$ to $\alpha$, this reveals that the method $R$ deems $\alpha$ a stronger performance than $\beta$. In that case, a switch from $\beta$ to $\alpha$ should never deteriorate $i$’s ranking, and, at any profile where $i$ is
Separability when \( i \) hold, we obtain a contradiction to Separability when \( i \).

Step 2. \( (ii) \) that \( R \) is relexive and symmetric, and \( \) is complete.

These facts follow directly from the definition of \( \) and \( \); they do not rely on the assumptions that \( R \) is separable and consistent.

Step 2. For all \((i, \alpha), (j, \beta) \in X\), exactly one of the following statements holds: (i) \((i, \alpha) \succ (j, \beta)\), (ii) \((j, \beta) \succ (i, \alpha)\), (iii) \((i, \alpha) \sim (j, \beta)\).

Let \((i, \alpha), (j, \beta) \in X\). By Step 1, at least one of statements (i), (ii), (iii) holds. If both (i) and (ii) hold, we obtain a contradiction to Separability when \( i \neq j \) and a contradiction to Consistency when \( i = j \). If both (i) and (iii) hold, or if both (ii) and (iii) hold, we obtain a contradiction to Separability when \( i \neq j \) and an immediate contradiction when \( i = j \).
Because of Steps 1 and 2, in order to prove that $\succsim$ is an ordering, it is enough to show that $\succ$ and $\sim$ are transitive. These are the next two steps.

**Step 3.** $\succ$ is transitive.

Let $(i_1, \alpha), (i_2, \beta), (i_3, \gamma) \in X$ be such that $(i_1, \alpha) \succ (i_2, \beta) \succ (i_3, \gamma)$.

**Case 1.** $i_1 = i_2 = i_3$, say, $(1, \alpha) \succ (1, \beta) \succ (1, \gamma)$.

Then there exist $a_{-1}, b_{-1} \in A_{-1}$ and $i, j \in N \setminus \{1\}$ such that

$$1P(\alpha, a_{-1})iR(\beta, a_{-1})1 \text{ or } 1R(\alpha, a_{-1})iP(\beta, a_{-1})1$$

(1)

and

$$1P(\beta, b_{-1})jR(\gamma, b_{-1})1 \text{ or } 1R(\beta, b_{-1})jP(\gamma, b_{-1})1.$$  

(2)

In order to prove $(1, \alpha) \succ (1, \gamma)$, we show that neither $(1, \gamma) \succ (1, \alpha)$ nor $(1, \alpha) \sim (1, \gamma)$.

If $(1, \gamma) \succ (1, \alpha)$, then there exists $c_{-1} \in A_{-1}$ and $k \in N \setminus \{1\}$ such that

$$1P(\gamma, c_{-1})kR(\alpha, c_{-1})1 \text{ or } 1R(\gamma, c_{-1})kP(\alpha, c_{-1})1.$$  

(3)

If $1R(\beta, c_{-1})k$, (3) implies $1R(\beta, c_{-1})kR(\alpha, c_{-1})1$ which, combined with (1), contradicts Consistency. If $kR(\beta, c_{-1})1$, (3) implies $1R(\gamma, c_{-1})kR(\beta, c_{-1})1$ which, combined with (2), contradicts Consistency again.

If $(1, \alpha) \sim (1, \gamma)$, then for all $x_{-1} \in A_{-1}$ and all $k \in N \setminus \{1\}$, we have $1R(\alpha, x_{-1})k \iff 1R(\gamma, x_{-1})k$. Specializing this equivalence to $k = i, x_{-1} = a_{-1}$, and combining it with statement (1) yields $1P(\gamma, a_{-1})iR(\beta, a_{-1})1$ or $1R(\gamma, a_{-1})iP(\beta, a_{-1})1$. This statement, in conjunction with (2), contradicts Consistency again.

**Case 2.** $i_1 = i_2 \neq i_3$, say, $(1, \alpha) \succ (1, \beta) \succ (2, \gamma)$.

Then there exist $a_{-1} \in A_{-1}$ and $i \in N \setminus \{1\}$ such that (1) holds and there exists $b_{-12} \in A_{-12}$ such that

$$1P(\beta, \gamma, b_{-12})2.$$  

(4)

By Consistency, (1) implies that there is no $c_{-1} \in A_{-1}$ such that $1R(\beta, c_{-1})2R(\alpha, c_{-1})1$. Therefore (4) implies $1P(\alpha, \gamma, b_{-12})2$, hence, $(1, \alpha) \succ (2, \gamma)$.

**Case 3.** $i_1 = i_3 \neq i_2$, say, $(1, \alpha) \succ (2, \beta) \succ (1, \gamma)$.

Then there exist $a_{-12}, b_{-12} \in A_{-12}$ such that $1P(\alpha, \beta, a_{-12})2P(\gamma, \beta, b_{-12})1$, hence, by Separability, $1P(\alpha, \beta, a_{-12})2P(\gamma, \beta, a_{-12})1$, which implies $(1, \alpha) \succ (1, \gamma)$.

**Case 4.** $i_1 \neq i_2 = i_3$, say, $(1, \alpha) \succ (2, \beta) \succ (2, \gamma)$.

Then there exists $a_{-12} \in A_{-12}$ such that

$$1P(\alpha, \beta, a_{-12})2.$$  

(5)
and there exist \( b_{-2} \in A_{-2} \) and \( i \in N \setminus \{2\} \) such that

\[
2P(\beta, b_{-2})iR(\gamma, b_{-2})2 \text{ or } 2R(\beta, b_{-2})iP(\gamma, b_{-2})2.
\] (6)

By Consistency, (6) implies that there is no \( c_{-2} \in A_{-2} \) such that \( 2R(\gamma, c_{-2})1R(\beta, c_{-2})2 \). Therefore (5) implies \( 1P(\alpha, \gamma, a_{-12})2 \), hence, \( (1, \alpha) \succ (2, \gamma) \).

**Case 5.** \( i_1, i_2, i_3 \) are all distinct, say, \( (1, \alpha) \succ (2, \beta) \succ (3, \gamma) \).

Then there exist \( a_{-12} \in A_{-12} \) and \( b_{-23} \in A_{-23} \) such that

\[
1P(\alpha, \beta, a_{-12})2P(\beta, \gamma, b_{-23})3.
\] (7)

Let \( (\alpha, \beta, \gamma, a_{-123}) \) be the profile obtained by replacing item 3’s performance in \( (\alpha, \beta, a_{-12}) \) with \( \gamma \). By Separability, (7) implies \( 1P(\alpha, \beta, \gamma, a_{-123})2P(\alpha, \beta, \gamma, a_{-123})3 \), hence, \( 1P(\alpha, \beta, \gamma, a_{-123})3 \) (because \( P(\alpha, \beta, \gamma, a_{-123}) \) is transitive), and therefore \( (1, \alpha) \succ (3, \gamma) \).

**Step 4.** \( \sim \) is transitive.

Let \( (i_1, \alpha), (i_2, \beta), (i_3, \gamma) \in X \) be such that \( (i_1, \alpha) \sim (i_2, \beta) \sim (i_3, \gamma) \).

**Case 1.** \( i_1 = i_2 = i_3 \), say, \( (1, \alpha) \sim (1, \beta) \sim (1, \gamma) \).

Then for all \( a_{-1} \in A_{-1} \) and all \( i \in N \setminus \{1\} \), \( 1R(\alpha, a_{-1})i \leftrightarrow 1R(\beta, a_{-1})i \leftrightarrow 1R(\gamma, a_{-1})i \). Therefore \( 1R(\alpha, a_{-1})i \leftrightarrow 1R(\gamma, a_{-1})i \), hence, \( (1, \alpha) \sim (1, \gamma) \).

**Case 2.** \( i_1 = i_2 \neq i_3 \), say, \( (1, \alpha) \sim (1, \beta) \sim (2, \gamma) \).

Then for all \( a_{-1} \in A_{-1} \) and all \( i \in N \setminus \{1\} \),

\[
1R(\alpha, a_{-1})i \leftrightarrow 1R(\beta, a_{-1})i
\] (8)

and there exists \( b_{-12} \in A_{-12} \) such that

\[
1I(\beta, \gamma, b_{-12})2.
\] (9)

Applying (8) with \( a_{-1} = (\gamma, b_{-12}) \) and \( i = 2 \) yields \( 1R(\alpha, \gamma, b_{-12})2 \leftrightarrow 1R(\beta, \gamma, b_{-12})2 \). Combining this with (9) implies \( 1I(\alpha, \gamma, b_{-12})2 \), hence, \( (1, \alpha) \sim (2, \gamma) \).

**Case 3.** \( i_1 = i_3 \neq i_2 \), say, \( (1, \alpha) \sim (2, \beta) \sim (1, \gamma) \).

Then there exist \( a_{-12}, b_{-12} \in A_{-12} \) such that \( 1I(\alpha, \beta, a_{-12})2I(\gamma, \beta, b_{-12})1 \), hence, by Separability, \( 1I(\alpha, \beta, a_{-12})2I(\gamma, \beta, a_{-12})1 \), which implies

\[
1R(\gamma, \beta, a_{-12})2R(\alpha, \beta, a_{-12})1.
\] (10)

To prove \( (1, \alpha) \sim (1, \gamma) \), we must show that \( 1R(\alpha, c_{-1})i \leftrightarrow 1R(\gamma, c_{-1})i \) for all \( c_{-1} \in A_{-1} \) and all \( i \in N \setminus \{1\} \). Suppose, on the contrary, that there exist \( c_{-1} \in A_{-1} \) and \( i \in N \setminus \{1\} \) such that
1P(α, c−1)iR(γ, c−1)1 or 1R(α, c−1)iP(γ, c−1)1. Then, by Consistency, there is no d−1 ∈ A−1 such that 1R(γ, d−1)2R(α, d−1)1, contradicting (10).

**Case 4.** i1 ≠ i2 = i3, say, (1, α) ∼ (2, β) ∼ (2, γ).

Then there exist a−12 ∈ A−12 such that

\[ 1I(α, β, a_{−12})2 \] (11)

and, for all b−2 ∈ A−2 and all i ∈ N \ {2}, we have 2R(β, b−2)i ↔ 2R(γ, b−2)i. Applying this equivalence with b−2 = (α, a−12) and i = 1, we get 2R(α, β, a−12)1 ↔ 2R(α, γ, a−12)1. Combining this with (11) yields 1I(α, γ, a−12)2, hence, (1, α) ∼ (2, γ).

**Case 5.** i1, i2, i3 are all distinct, say, (1, α) ∼ (2, β) ∼ (3, γ).

Then there exist a−12 ∈ A−12 and b−23 ∈ A−23 such that 1I(α, β, a−12)2I(β, γ, b−23)3. By Separability, this implies 1I(α, β, γ, a−123)2I(α, β, γ, a−123)3, hence, 1P(α, β, γ, a−123)3 (because I(α, β, γ, a−123) is transitive), and therefore (1, α) ∼ (3, γ).

This completes the proof that ∼ is an ordering.

**Step 5.** It remains to check that iR(a)j ⇔ (i, a) ∼ (j, a) for all i, j ∈ N and all a ∈ AN.

Fix i, j ∈ N and a ∈ AN. The case i = j being trivial, assume i ≠ j. It follows directly from the definition of ∼ and ~ that

\[ iP(a)j ⇒ (i, a) ∼ (j, a) \] and \[ iI(a)j ⇒ (i, a) ∼ (j, a), \]

hence, iR(a)j ⇒ (i, a) ∼ (j, a). The converse implication follows from Step 2 and the completeness of R(a).\]

As a direct corollary to Lemma 1, we obtain the central result of this section.

**Theorem 1.** Suppose that A1, ..., An are finite. A ranking method R : AN → RN is separable and consistent if and only if it is a ranking-by-rating method.

**Proof.** If there exist functions v1 ∈ R^A1, ..., vn ∈ R^An such that iR(a)j ⇐⇒ vi(a_i) ≥ v_j(a_j) for all i, j ∈ N and all a ∈ AN, it is straightforward to check that the method R is separable and consistent.

Conversely, if R is separable and consistent, Lemma 1 guarantees that there exists an ordering ∼ on X := {(i, α) : i ∈ N and α ∈ A_i} such that iR(a)j ⇐⇒ (i, a_i) ∼ (j, a_j) for all i, j ∈ N and a ∈ A_N. Because X is finite, the ordering ∼ admits a numerical representation: there exists a function V : X → R such that V(i, a_i) ≥ V(j, a_j) ⇐⇒ (i, a_i) ∼ (j, a_j). If, for each i ∈ N, we define the function v_i : A_i → R by v_i(a_i) = V(i, a_i), then iR(a)j ⇐⇒ v_i(a_i) ≥ v_j(a_j) for all i, j ∈ N and all a ∈ AN, proving that R is a ranking-by-rating method.\]
The finiteness assumption in Theorem 1 is used to ensure the representability of the ordering \( \preceq \) but the result is easily adapted to the infinite case. Assume that \( A_N \) is a perfectly separable topological space\(^3\) and suppose that the ranking method is continuous in the sense that any strict ordering of two items is robust to small changes in the performance profile.

**Continuity.** For all \( i, j \in N \), the set \( \{ a \in A_N \mid iP(a)j \} \) is relatively open in \( A_N \).

A straightforward application of Theorem II in Debreu (1954), which we omit, then delivers the following corollary to Theorem 1.

**Corollary 1.** If \( A_N \) is a perfectly separable topological space, then a ranking method \( R : A_N \to \mathcal{R}_N \) is separable, consistent, and continuous if and only if there exist continuous functions \( v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n} \) such that \( iR(a)j \iff v_i(a_i) \geq v_j(a_j) \) for all \( i, j \in N \) and all \( a \in A_N \).

## 5 Symmetry and Monotonicity

Consistency may seem complicated and somewhat contrived. The current section identifies two simpler conditions, each of which guarantees that a separable method is consistent. Both arise naturally in particular cases of our model. The first condition, Symmetry, applies when the performance sets of all items coincide, that is, when \( A_1 = \ldots = A_n = A \). In that case, \( A_N = A^N \), and we call a method \( R : A^N \to \mathcal{R}_N \) symmetric if permuting the performances of two items results in permuting their positions in the ranking.

**Symmetry.** For all \( i, j \in N \), all \( a \in A^N \), and every bijection \( \pi \) from \( N \) to \( N \), \( iR(a)j \iff \pi(i)R(\pi(a))\pi(j) \), where \( \pi a \) is the performance profile defined by \((\pi a)_{\pi(i)} = a_i \) for all \( i \in N \).

The second condition, Monotonicity, applies when each performance set \( A_i \) is endowed with a linear ordering \( \succeq_\cdot \). We call a ranking method monotonic if a higher performance improves an item’s position in the ranking.

**Monotonicity.** For all distinct \( i, j \in N \) and \( a, a' \in A_N \), \( [iR(a)j \text{ and } a'_i >_i a_i] \Rightarrow [iP(a'_i, a_{-i})j] \).

This is a strict form of monotonicity. Note that it implies \([iR(a)j \text{ and } a_j >_j a'_j] \Rightarrow [iP(a'_j, a_{-j})j]\) for all distinct \( i, j \in N \) and \( a, a' \in A_N \).

The crucial observation is recorded in the following lemma.

**Lemma 2.** Let \( R : A_N \to \mathcal{R}_N \) be a separable ranking method. Suppose that (i) \( A_1 = \ldots = A_n = A \) and \( R \) is symmetric, or (ii) \( A_1, \ldots, A_n \) are linearly ordered and \( R \) is monotonic. Then \( R \) is consistent.

---

\(^3\)This means that there exists a countable class of open sets such that every open set in \( A_N \) is a union of sets in that class.
Proof. Let $R : A_N \to R_N$ be a separable ranking method.

(1) Suppose first that $A_1 = \ldots = A_n = A$ and that $R : A_N = A^N \to R_N$ is symmetric. If $R$ is not consistent, we may assume without loss of generality that there exist $\alpha, \beta \in A$, $a, b \in A^N$, and $i \in N \setminus \{1\}$ such that

$$1P(\alpha, a_{-1})2R(\beta, a_{-1})1 \text{ or } 1R(\alpha, a_{-1})2P(\beta, a_{-1})1$$

and

$$1R(\beta, b_{-1})iR(\alpha, b_{-1})1.$$  \hfill (13)

**Case 1.** $i \neq 2$.

From (12) and Separability,

$$1P(\alpha, a_2, b_{-12})2R(\beta, a_2, b_{-12})1 \text{ or } 1R(\alpha, a_2, b_{-12})2P(\beta, a_2, b_{-12})1.$$  \hfill (14)

From (13) and Separability,

$$1R(\beta, a_2, b_{-12})iR(\alpha, a_2, b_{-12})1.$$  \hfill (15)

Because $R(\alpha, a_2, b_{-12})$ and $R(\beta, a_2, b_{-12})$ are transitive, (14) and (15) imply, after some manipulations,

$$iP(\alpha, a_2, b_{-12})2R(\beta, a_2, b_{-12})i \text{ or } iR(\alpha, a_2, b_{-12})2P(\beta, a_2, b_{-12})i,$$

which contradicts Separability.

**Case 2.** $i = 2$.

From (12) and Separability,

$$1P(\alpha, a_2, b_2, b_{-123})2R(\beta, a_2, b_2, b_{-123})1 \text{ or } 1R(\alpha, a_2, b_2, b_{-123})2P(\beta, a_2, b_2, b_{-123})1.$$  \hfill (16)

From (13) and Separability – and since $i = 2$,

$$1R(\beta, a_2, b_2, b_{-123})2R(\alpha, a_2, b_2, b_{-123})1.$$

Exchanging the performances of items 2 and 3, Symmetry now implies

$$1R(\beta, a_2, b_2, b_{-123})3R(\alpha, a_2, b_2, b_{-123})1.$$  \hfill (17)

Because $R(\alpha, a_2, b_2, b_{-123})$ and $R(\beta, a_2, b_2, b_{-123})$ are transitive, (16) and (17) imply, after some manipulations,

$$3P(\alpha, a_2, b_2, b_{-123})2R(\beta, a_2, b_2, b_{-123})3 \text{ or } 3R(\alpha, a_2, b_2, b_{-123})2P(\beta, a_2, b_2, b_{-123})3.$$
which contradicts Separability again.

(2) Suppose next that each performance set $A_i$ is endowed with a linear order $\geq_i$ and that $R : A_N \rightarrow \mathcal{R}_N$ is monotonic. This case is straightforward. If $R$ is not consistent, we may assume without loss of generality that there exist $\alpha, \beta \in A_1$, $a, b \in A_N$, and $i \in N \setminus \{1\}$ such that (12) and (13) hold. Clearly, $\alpha \neq \beta$. Since $\geq_1$ is a linear ordering, either $\alpha >_1 \beta$ or $\beta >_1 \alpha$. If $\alpha >_1 \beta$, then (13) contradicts Monotonicity. If $\beta >_1 \alpha$, then (12) contradicts Monotonicity.

Notice that Separability is not used in part (2) of the proof: we have, in fact, proved that every monotonic method, separable or not, is consistent.

In the finite case, Theorem 1 and Lemma 2 imply that a separable method which is symmetric or monotonic is a ranking-by-rating method. Of course, Symmetry and Monotonicity translate into restrictions on the rating functions: under Symmetry these functions must all coincide, under Monotonicity they must be increasing. When the performance sets are infinite, Corollary 1 and Lemma 2 deliver a similar representation with continuous rating functions. These results are recorded below; we omit their obvious proofs.

**Theorem 2.**

(i) If $A_1 = \ldots = A_n = A$ is a finite set, then a ranking method $R : A^N \rightarrow \mathcal{R}_N$ is separable and symmetric if and only if there exists a function $v : A \rightarrow \mathbb{R}$ such that $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ for all $i, j \in N$ and all $a \in A^N$.

(ii) If $A_1, \ldots, A_n$ are linearly ordered finite sets, then a ranking method $R : A_N \rightarrow \mathcal{R}_N$ is separable and monotonic if and only if there exist increasing functions $v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$.

**Corollary 2.**

(i) If $A_1 = \ldots = A_n = A$ is a perfectly separable topological space, then a ranking method $R : A^N \rightarrow \mathcal{R}_N$ is separable, symmetric, and continuous if and only if there exists a continuous function $v : A \rightarrow \mathbb{R}$ such that $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ for all $i, j \in N$ and all $a \in A^N$.

(ii) If $A_N$ is a perfectly separable topological space and each of $A_1, \ldots, A_n$ is endowed with a linear order, then a ranking method $R : A_N \rightarrow \mathcal{R}_N$ is separable, monotonic, and continuous if and only if there exist increasing and continuous functions $v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$.

In statement (ii) of Corollary 2, each item’s performance set is assumed to be completely ordered. The result does not extend to the case where these sets are only partially ordered and the method is assumed to be monotonic with respect to that partial order.

**Example 2.** Let $N = \{1, 2, 3\}$, $A_i = A = [0, 1]^2$ for all $i \in N$. Endow $A$ with the usual partial order $\geq$. A generic performance profile is a vector $a = (a_1, a_2, a_3) = ((a_1^1, a_1^2), (a_2^1, a_2^2), (a_3^1, a_3^2)) \in \mathbb{R}^{A_i}$.
Define the functions \( w_1, w_2, w_3 \) from \( A^{1,2,3} \) to \( \mathbb{R} \) by

\[
\begin{align*}
    w_1(a) &= (1 - a_3^2)a_1^1 + (1 - a_2^1)a_2^2, \\
    w_2(a) &= \frac{1}{2}a_2^1 + \frac{1}{2}a_2^2, \\
    w_3(a) &= -1
\end{align*}
\]

for all \( a \in A^{1,2,3} \). Note that \( w_1(a) \) varies with \( a_2 \). Define the method \( R \) by

\[
iR(a)j \iff w_i(a) \geq w_j(a)
\]

for all \( a \in A^{1,2,3} \) and all \( i, j \in \{1, 2, 3\} \). Since \( w_1(a), w_2(a) \geq 0 \) for all \( a \), item 3 is ranked last at every performance profile. Moreover, the ranking of items 1 and 2 does not change with 3’s performance. So \( R \) is separable. Since \( w_1, w_2, w_3 \) are continuous, \( R \) is also continuous. Furthermore, it is monotonic because \( w_1 \) is increasing in \( a_1 \) and \( w_2 \) is increasing in \( a_2 \).

This method is not a ranking-by-rating method. By definition of \( w_1, w_2, w_3 \) and \( R \),

\[
\begin{align*}
    1P((1, 0), (1, 0), (0, 0)) &= 2, \\
    2P((0, 1), (1, 0), (0, 0)) &= 1, \\
    1P((0, 1), (0, 1), (0, 0)) &= 2, \\
    2P((1, 0), (0, 1), (0, 0)) &= 1.
\end{align*}
\]

If \( v_1, v_2, v_3 \) were rating functions from \( A \) to \( \mathbb{R} \) such that \( iR(a)j \iff v_i(a_i) \geq v_j(a_j) \) for all \( a \in A^{1,2,3} \) and all \( i, j \in \{1, 2, 3\} \), then

\[
\begin{align*}
    v_1(1, 0) &> v_2(1, 0), \\
    v_2(1, 0) &> v_1(0, 1), \\
    v_1(0, 1) &> v_2(0, 1), \\
    v_2(0, 1) &> v_1(1, 0),
\end{align*}
\]

which are incompatible inequalities.

### 6 Invariance

This section studies the case where the performance sets of the items coincide and are endowed with a partial order structure. More precisely, we assume that there is a finite set of criteria \( M = \{1, \ldots, m\} \) and \( A_i = A = \mathbb{R}_+^M \) for each item \( i \in N \). A generic performance for item \( i \) is a vector \( a_i = (a_i^1, \ldots, a_i^m) \in A \). A performance profile is a matrix \( a = (a_i^h) \in A^N : \) rows correspond
to items, columns to criteria, and the number $a_i^h$ measures item $i$’s performance according to criterion $h$. We write $b_i > a_i$ if $b_i^h > a_i^h$ for all $h \in M$ and $b_i \neq a_i$. With a slight abuse of our earlier terminology, we now call $R$ monotonic if $[iR(a)j$ and $a_i' > a_i] \Rightarrow [iP(a_i',a_{-i})j]$ for all distinct $i,j \in N$ and $a,a' \in A_N$.

An important concern in this multidimensional framework is expressed by the condition of Invariance. The condition says that the ordering of the items should remain unchanged when their performances according to a given criterion are all multiplied by the same positive number. This is compelling if performances are measured on non-comparable scales across criteria.

To express the condition formally, we use the following notation. For every $\lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{R}^+^M$, let $dg(\lambda)$ denote the $m \times m$ diagonal matrix whose $h$th diagonal entry is $\lambda^h$. With this notation, $a \cdot dg(\lambda)$ is the performance matrix obtained by multiplying each column $h$ of $a$ by $\lambda^h$.

**Invariance.** For all $a \in A^N$ and $\lambda \in \mathbb{R}^+^M$, $R(a \cdot dg(\lambda)) = R(a)$.

Let $\Delta^M_{++}$ denote the relative interior of the unit simplex of $\mathbb{R}^M$.

**Theorem 3.** Let $A_1 = \ldots = A_n = A = \mathbb{R}^M$. A ranking method $R : A^N \rightarrow \mathcal{R}_N$ is separable, symmetric, monotonic, continuous, and invariant if and only if there exists $\beta = (\beta^1, \ldots, \beta^m) \in \Delta^M_{++}$ such that

$$iR(a)j \Leftrightarrow \prod_{h \in M} (a^h_i)^{\beta^h} \geq \prod_{h \in M} (a^h_j)^{\beta^h} \text{ for all } i,j \in N \text{ and all } a \in A^N.$$  \hspace{1cm} (18)

**Proof.** The “if” statement is clear. The proof of the “only if” statement is a straightforward consequence of Corollary 2 and Osborne’s (1976) characterization of the monotonic transformations of the weighted geometric means.

Fix a separable, symmetric, monotonic, continuous, and invariant method $R$. By statement (i) in Corollary 2, there exists a continuous function $w : \mathbb{R}^+^M \rightarrow \mathbb{R}$ such that

$$iR(a)j \Leftrightarrow w(a_i) \geq w(a_j) \text{ for all } i,j \in N \text{ and all } a \in A^N.$$  \hspace{1cm} (19)

Since $R$ is monotonic, $w$ is increasing: $a_i < a_j \Rightarrow w(a_i) < w(a_j)$. Because $R$ is invariant, $w$ is ordinally invariant in the sense that

$$w(a_i) \leq w(a_j) \Leftrightarrow w(\lambda^1 a^1_i, \ldots, \lambda^m a^m_i) \leq w(\lambda^1 a^1_j, \ldots, \lambda^m a^m_j)$$

for all $\lambda \in \mathbb{R}^+^M$. By Osborne (1976), there exist $\beta = (\beta^1, \ldots, \beta^m) \in \mathbb{R}^M$ and an increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$w(a_i) = f \left( \prod_{h \in M} (a^h_i)^{\beta^h} \right)$$  \hspace{1cm} (20)

for all $a_i \in A$. (In Osborne’s theorem, $w$ is nondecreasing and $\beta \in \mathbb{R}^M$. In our case, the fact that $w$ is increasing guarantees that $\beta \in \mathbb{R}^M_{++}$. The normalization $\beta \in \Delta^M_{++}$ is innocuous.) Statement
(18) now follows from (19) and (20).

Of course, the weighted geometric mean numerical representation in Theorem 3 is unique only up to an increasing transformation.

7 Separable Grading

We conclude with a defense of separability in the context of cardinal evaluation of multidimensional performances. From now on, for comparability with existing work, we restrict our attention to the set \( A_+^N \) of positive \( n \times m \) matrices. A grading method is a function \( G : A_+^N \rightarrow \Delta^N \), where \( \Delta^N \) denotes the unit simplex of \( \mathbb{R}^N \). The vector \( G(a) = (G_1(a), ..., G_n(a)) \) is the grade distribution assigned by the method \( G \) to the performance matrix \( a \). The grade of item \( i \), \( G_i(a) \), is interpreted as a cardinal measure of its multidimensional performance. A grading method \( G \) clearly induces a ranking method \( R_G \) defined on \( A_+^N \) by

\[
i R_G(a)j \iff G_i(a) \geq G_j(a),
\]

but the information contained in the grade distribution \( G(a) \) is richer than that in the induced ranking \( R_G(a) \). As mentioned in Footnote 1, grading methods are the traditional object of study in the literature on multidimensional performance evaluation. We call \( G \) ordinally separable if \( R_G \) is separable.

Assuming that performances are cardinally measurable on non-comparable scales, two properties of grading methods appear to be essential. The first is the cardinal version of the invariance axiom discussed earlier.

**Cardinal Invariance.** For all \( a \in A_+^N \) and \( \lambda \in \mathbb{R}_{++}^M \), \( G(a \cdot dg(\lambda)) = G(a) \).

The second condition is Homogeneity. It requires that if an item’s performances with respect to all criteria are multiplied by the same positive number, the ratio of that item’s grade to any other item’s grade is multiplied by the same number. This is compelling if performances with respect to each criterion are cardinally measurable. For every \( \mu = (\mu_1, ..., \mu_n) \in \mathbb{R}_{++}^N \), let \( dg(\mu) \) denote the \( n \times n \) diagonal matrix whose \( i \)th diagonal entry is \( \mu_i \). With this notation, \( dg(\mu) \cdot a \) is the performance matrix obtained by multiplying each row \( i \) of \( a \) by \( \mu_i \).

**Homogeneity.** For all \( a \in A_+^N \) and \( \mu \in \mathbb{R}_{++}^N \), \( G(dg(\mu) \cdot a) \) is proportional to \( dg(\mu) \cdot G(a) \).

Most popular grading methods fail at least one of these two axioms. In fact, Cardinal Invariance and Homogeneity together have far-reaching consequences. Call a performance matrix \( a \in A_+^N \) doubly balanced if \( \sum_{i \in N} a_i^h = 1 \) for all \( h \in M \) and \( \sum_{h \in M} a_i^h = m/n \) for all \( i \in N \). Let \( A_{++}^N \) denote the set of doubly balanced matrices. Sinkhorn (1967) proves that for every matrix \( a \in A_+^N \) there exist a unique vector \( \lambda(a) \in \mathbb{R}_{++}^M \) and a unique vector \( \mu(a) \in \mathbb{R}_{++}^N \) such that \( dg(\mu(a)) \cdot a \cdot dg(\lambda(a)) =: a^* \) is doubly balanced. This means that every positive matrix \( a \) can be reduced to a uniquely defined doubly balanced matrix \( a^* \) by rescaling its rows and columns. It follows that a cardinally invariant
and homogeneous grading method is completely determined by its behavior on the doubly balanced matrices. Formally,

**Corollary to Sinkhorn’s Theorem.** A grading method \( G : A^N_+ \to \Delta^N \) is cardinally invariant and homogeneous if and only if there exists a function \( G^* : A^N_* \to \Delta^N \) such that \( G(a) \) is proportional to \((dg(\mu(a)))^{-1} \cdot G^*(a^*)\) for all \( a \in A^N_+ \).

Building on this corollary, Demange (2014) pins down the handicap-based grading method by adding to Cardinal Invariance and Homogeneity the Uniformity axiom, which requires that all items should be tied when the performance matrix is doubly balanced.

**Uniformity.** For all \( a \in A^N_+ \), \( G(a) = (\frac{1}{n}, \ldots, \frac{1}{n}) \).

The handicap-based method is *not* ordinally separable. Its computation requires an iterative procedure, and the grades it yields vary with the performance matrix in ways that are difficult to apprehend.

It may therefore be worth pointing out that *geometric mean grading method*

\[
G^gm_i(a) = \frac{\prod_{h \in M} (a^h_i)^{\frac{1}{n}}}{\sum_{j \in N} \prod_{h \in M} (a^h_j)^{\frac{1}{n}}},
\]

is cardinally invariant, homogeneous, and ordinally separable. Thus, while Cardinal Invariance, Homogeneity, Ordinal Separability, and Uniformity are incompatible, the first three of these axioms are not. If the first two are considered a must, we are left with a choice between the last two. Uniformity amounts to imposing the *arithmetic* mean criterion on the doubly balanced matrices. This creates an obvious tension with Cardinal Invariance and is probably not compelling. Ranking item 1 above 2 and 3 for the matrix

\[
\begin{pmatrix}
3/9 & 3/9 & 3/9 \\
2/9 & 3/9 & 4/9 \\
4/9 & 3/9 & 2/9
\end{pmatrix},
\]

as the geometric mean does, seems to be reasonable and is supported by an argument of variability aversion: the fact that the scores of item 1 coincide on all criteria gives them added value. Separability may well be worth sacrificing Uniformity.\(^5\)

\(^4\)In fact, Cardinal Invariance, Homogeneity, and the requirement that an item’s grade should be the geometric mean of its performances when the performance matrix is doubly balanced, together pin down \( G^gm \), thereby implying Ordinal Separability: this is again a consequence of the above corollary to Sinkhorn’s theorem.

\(^5\)The method \( G^gm \) is also ordinally monotonic, in the sense that \( R_{G^gm} \) is monotonic. The handicap-based method has not been shown to be ordinally monotonic.
8 References


Arrow, K.J. (1963), Social choice and individual values, 2nd edition, New York: Wiley.


Figure 1. A full-range separable method not of the ranking-by-rating type