Fairness and externalities

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Abstract

We study equitable allocation of indivisible goods and money among agents with other-regarding preferences. First, we argue that Foley's (1967) equity test, i.e., the requirement that no agent prefer the allocation obtained by swapping her consumption with another agent, is suitable for our environment. Then, we establish the existence of allocations passing this test for a general domain of preferences that accommodates prominent other-regarding preferences. Our results are relevant for equitable allocation among inequity-averse agents and in a domain with linear externalities that we introduce. Finally, we present conditions guaranteeing that these allocations are efficient.

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1 Introduction

The need to equitably and efficiently allocate commonly and symmetrically owned resources arises naturally in a market economy. Many economic activities that require agents’ cooperation result in the division of assets, gains, losses, etc. Due to diversity of preferences, it is not trivial what an equitable distribution is. Foley (1967) advanced in this endeavor and developed the following equity test: “ask each person to imagine changing places with every other [...] If no one is willing to change, the allocation is equitable.”1 An extensive literature has investigated the existence of allocations satisfying this property and its compatibility with efficiency (see Thomson, 2010, for a survey). A common feature of these studies is that each agent’s welfare is assumed to be independent of the consumption of the other agents. Under this

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1Kolm (1971) refers to an earlier formulation of this test by Tinbergen (1946).
assumption, Foley's test is referred to as the "no-envy" test, and can be simplified as the requirement that no agent prefer the consumption of any other agent to her own consumption (Varian, 1974). Assuming externalities away entails a significant loss of generality. A growing literature in experimental economics has consistently documented human behavior that cannot be rationalized by self-regarding preferences, but can be rationalized by other-regarding preferences (see Fehr and Schmidt, 2001, for a survey). This experimental evidence validates our everyday observation of altruism, philanthropy, conspicuous consumption, and the unavoidable partially public nature of consumption. We address this issue and include other-regarding preferences in the study of equitable allocation of indivisible goods and money. First, we argue that Foley's equity test is suitable and meaningful in our environment. Then, we establish the existence of allocations passing this test for a general domain of preferences that accommodates prominent other-regarding preferences. Our results are relevant for equitable allocation among inequity-averse agents (Fehr and Schmidt, 1999) and in a domain with linear externalities that we introduce. Finally, we present conditions guaranteeing these allocations are efficient.

We focus on assignment problems with money. That is, there are $n$ agents and $n$ objects, each agent must receive one object, and individual consumptions of money should add up to a given budget. Examples are the assignment of tasks and salaries among employees, the allocation of a bequest with indivisible goods, and the allocation of rooms and rent among roommates (Svensson, 1983; Maskin, 1987; Alkan et al., 1991). We depart from the assumption, universal in previous literature, that each agent’s preferences are defined on her consumption space. We consider, instead, agents who have preferences on the set of possible allocations. Our basic assumptions on preferences are continuity and anonymity of externalities, i.e., the welfare of each agent is not affected by a permutation of the allotments among the other agents. Our results may not hold if these assumptions are not satisfied (Sec. 3).  

Foley's equity test is applicable in our environment. More importantly, the normative support that has been recognized for this test when applied to economies with self-regarding preferences, is preserved in economies with anonymous other-regarding preferences. Essentially, it requires no interpersonal comparisons of utility and implements the principles of symmetry (Varian, 1974) and equal opportunity (Kolm, 1971). In Sec. 2 we discuss these properties at length. Our analysis there gives us two further insights. First, we discover other situations, beyond equitable allocation of commonly owned resources, for which allocations passing Foley's test are desirable, e.g., the allocation of tasks and salary among employees. Second, it

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2Our model is motivated by other-regarding preferences exhibited by agents who face distributive scenarios with anonymous partners as in most laboratory experiments and charitable giving. In particular, these agents do not have cardinal information about the welfare of the other agents. Thus, we concentrate on consumption externalities.
gives us a basis to evaluate alternative extensions of the no-envy test to economies with other-regarding preferences. We conclude that we should favor Foley’s original formulation. Here we also observe that, in our environment, the allocations passing Foley’s test coincide with those obtained by an equal income market, which is an intuitively equitable institution (Varian, 1976). Thus, by studying the existence and welfare properties of these allocations, we are also studying the existence and welfare properties of equal income market outcomes. We formalize the connection between the two approaches in Sec. 5.1. To avoid confusion with the multiple alternative equity tests that have been proposed in the literature, we refer to the allocations passing Foley’s test as being “non-contestable (on equity grounds).”

For simplicity of the presentation, we assume, for most of the paper, that consumptions of money are required to be non-negative. This is also a realistic restriction in some environments as the allocation of tasks and salary. Our basic existence result, Theorem 1, states that if the preferences of at least \( n - 1 \) agents satisfy a requirement that we refer to as the “compensation assumption,” non-contestable allocations exist. Obviously, if the available budget cannot compensate the agents for not receiving one object or the other, non-contestable allocations may not exist. This can happen for two reasons. First, the agents do not care enough about their private consumption of money, compared to the object they receive. Second, even if agents care about money sufficiently, there might not be enough money. These are exactly the situations that our compensation assumption excludes.

We investigate the implications of our basic existence result for two domains of other-regarding preferences of interest. First, we consider inequity-averse agents, i.e., agents who not only care about their own consumption, but also care about the equitability of the profile of consumptions. Such an agent subjectively assesses equity of an allocation. Her ideal is equality of consumption value. She loses welfare when, ceteris paribus, her perception of the value of the consumptions of others deviates from the value of her own consumption. Our domain is parametrized by the agent’s value functions, a coefficient that captures the agent’s aversion to inequity against herself, and a coefficient that captures aversion to inequity against others as in Fehr and Schmidt (1999). In our domain, an agent’s perception of the value of the allotment of others may depend on her own allotment. In particular, this can accommodate an agent’s endowment effect, i.e., her propensity to assign a different

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3 "Envy" and “inequity” are commonly understood as consumption externalities that can be captured in our model. Referring to the allocations that pass our test as “fair,” “envy-free,” or “equitable” could mislead the reader into thinking that these allocations minimize certain externality that the agents internalize in their preferences. Additionally, referring to these allocations as “swap-proof” may suggest a strategic role of all the agents involved in the swap. We adopt the more neutral terminology of “non-contestability,” which suggests the application of an objective test of equity (see Kolm, 1971, 1995, for a related discussion; see Rawls, 1972, for a discussion on why distributive justice is not the absence of envy, understood as the pain at the well being of others).

4 Sec. 5.2 extends our results to environments where consumptions of money can be negative.
value to a bundle of goods when it is assigned to her, compared to when it is assigned to others (Kahneman et al., 1990). It turns out that when the endowment effect is bounded for inequity-averse agents and the budget is large enough, there are non-contestable allocations (Theorem 2) —this implies that under a mild restriction on inequity aversion, the agents’ subjective assessment of equity does not compromise the existence of allocations satisfying our objective equity criterion. Second, we introduce a domain of preferences for which externalities are a linear function of the consumption of money of the other agents. Here, non-contestable allocations exist, for a sufficiently large budget, whenever externality coefficients are uniformly bounded so that the direct effect of one dollar allocated to an agent is greater than its aggregate effect on the other agents (Theorem 4).

Besides symmetry and equal opportunity, another important dimension of distributive justice is economic efficiency (Moulin, 2003). It turns out that non-contestability may be incompatible with efficiency (Example 2 in Appendix). For this incompatibility to hold, there must be an aggregate imbalance between private and social interests, however. Indeed, our results reveal that for a wide range of externalities non-contestable allocations are always efficient —thus, our results can be interpreted as First Welfare Theorems for equal income competitive allocations. For inequity-averse agents, non-contestable allocations are efficient for an open set of parameters that depend on the bounds on the marginal value of money for the agents. This set grows monotonically and continuously when the bounds get closer for each agent. In the limit, where the marginal value of money is constant and equal for all value functions for each agent, there are no restrictions on aversion to inequity against the agent herself, and for large $n$, there are few restrictions on the aversion to inequity against others (Theorem 3). Finally, the conditions under which we guarantee that non-contestable allocations exist for linear externality preferences, also guarantee that each non-contestable allocation is efficient (Theorem 4).

Our existence results and welfare analysis provide a solid foundation for a theory of equitable allocation of objects and money with other-regarding preferences. Here it is also relevant to ask about the incentives issues associated with the implementation of non-contestable and efficient allocations (c.f., Tadenuma and Thomson, 1995a); about the manipulability of rules selecting these allocations (c.f., Andersson et al., 2014); about the refinement of this set based on further normative and practical requirements (c.f., Tadenuma and Thomson, 1995b); about the existence of systematic ways of selecting non-contestable and efficient allocations that implement the notion of solidarity requiring that when the budget increases all agents be better off (c.f., Alkan et al., 1991; Velez, 2011). These are all open questions, which are beyond the scope of this paper and are only meaningful now that we established general existence of non-contestable and efficient allocations for a rich domain of admissible other-regarding preferences.

Our existence results and welfare analysis generalize previous literature on exis-
istence of envy-free allocations for assignment problems with no externalities (Svensson, 1983; Maskin, 1987; Alkan et al., 1991). In particular, our inequity-averse domain contains the domain of preferences with no externalities considered in these previous studies. At a technical level, our results are related to the literature on land division (Dubins and Spanier, 1961). Part of our contribution is to show how the combinatorial approach to land division introduced by Su (1999) is suitable to handle other-regarding preferences in our model (Sec. 3).

Our work joins the efforts to solve the technical and conceptual challenges that arise in economic models when one accounts for the possibility that agents may care about the consumption of others, e.g., Gale (1984) for general equilibrium in a market with indivisibilities and price externalities; Nogushi and Zame (2006) for general equilibrium with a continuum of agents; Dufwenberg et al. (2011) for general equilibrium in classical economies; Maniquet and Moreno-Ternero (2012) for division of a bundle of infinitely divisible commodities; Brânzei et al. (2013) for cake cutting procedures; and Azrieli and Shmaya (2014) for the allocation of indivisible goods that can be shared by several agents. We discuss in detail the relevant papers to us, i.e., Gale (1984), Dufwenberg et al. (2011), and Azrieli and Shmaya (2014), in Sec. 3 and 5.

The remainder of the paper is organized as follows. Sec. 2 introduces our model. Sec. 3 presents and discusses our basic existence result. Sec. 4 presents our welfare analysis and the application of our existence results in domains of other-regarding preferences of interest. Sec. 5 discusses the relevance of our results for general equilibrium theory and the extension of our results when one allows negative consumptions of money and a different number of agents and objects. The Appendix collects all proofs.

## 2 Model

We consider the problem of allocating a social endowment of $n$ objects $A \equiv \{\alpha, \beta, \ldots\}$ and an amount, $M \in \mathbb{R}$, of an infinitely divisible good, which we refer to as “money,” among a group of $n$ agents $N \equiv \{1, 2, \ldots, n\}$. We assume that each agent should receive a bundle of one object and an amount of money. Agent $i$’s generic consumption is $(x_i, \mu_i)$, where $x_i \in \mathbb{R}$ and $\mu_i \in A$. An allocation is a pair $z \equiv (x, \mu)$, where $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n$ and $\mu \equiv (\mu_i)_{i \in N} \in A^n$, for which no two agents receive the same

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5The domain of profiles with no externalities for which we guarantee existence of non-contestable allocations in Theorem 1, properly contains the domain in which Svensson (1983, 1987); Maskin (1987); and Alkan et al. (1991) prove their existence results. First, we do not assume that an additional small amount of money necessarily makes the agent better off. Moreover, in Theorems 1, 2, and 6, we are able to waive the compensation assumption requirement for one agent.
object, i.e., for \( i \neq j, \mu_i \neq \mu_j \); \( z \) is feasible if \( \sum_{i \in N} x_i = M \).\(^6\) Alternatively, we write \( z \equiv (z_i)_{i \in N} \), where \( z_i \equiv (x_i, \mu_i) \) is agent \( i \)'s allotment at \( z \). We denote the consumption of money of the agent who receives object \( a \) at \( z \) by \( x_a \). We refer to \( x_a \) as the consumption of money associated with object \( a \) at \( z \). We denote the set of allocations by \( Z \), the set of feasible allocations by \( Z_M \), and the set of feasible allocations at which each agent’s consumption of money is non-negative by \( Z^{+}_M \). We endow \( Z \) with its natural box topology.\(^7\) It will be convenient for our purposes to reshuffle consumptions at a given allocation. That is, given \( z \equiv (z_i)_{i \in N} \in Z \) and permutation \( \pi : N \rightarrow N \), the allocation \( z_{\pi(i)} \) is that at which agent \( i \) receives bundle \( z_{\pi(i)} \).

Agents have complete, transitive, and continuous preferences on \( Z \).\(^8\) Given our continuity assumption, we introduce, without loss of generality, preference representations. Each agent is endowed with a continuous utility function on \( Z \). Agent \( i \)'s utility is \( u_i : Z \rightarrow \mathbb{R} \) and the utility profile is \( u \equiv (u_i)_{i \in N} \). We assume that externalities are anonymous, i.e., an agent’s welfare is not affected by reshuffling the consumption bundles of the other agents. Formally, for \( i \in N \) and \( z \in Z \), if \( \pi : N \rightarrow N \) is a permutation such that \( \pi(i) = i \), then \( u_i(z) = u_i(z_{\pi}) \). We denote the set of utility functions by \( \mathcal{U} \) and the set of profiles by \( \mathcal{U}^n \).

We investigate the existence and welfare properties of equitable allocations. Our welfare analysis is standard. An allocation \( z \in Z_M \) is efficient for \( u \) if there is no other feasible allocation that each agent weakly prefers to \( z \), and at least one agent strictly prefers to \( z \).

**Definition:** \( z \in Z \) is non-contestable on equity grounds for \( u \) (non-contestable for short) if no agent prefers the allocation obtained by swapping her consumption with any other agent.

Our equity test was originally proposed by Foley (1967) and since Varian (1974) has been specialized to environments without externalities. In this restricted environment it has been a central equity criterion (Thomson, 2010). In what follows, we argue that the normative support that has been recognized for the test when there are no externalities, is preserved when there are anonymous externalities.

First, the test is simple and avoids controversial interpersonal comparisons of utility. Second, it filters out the allocations that exhibit the clear-cut bias that one identifies when an agent is better off when she exchanges her place with another agent (Varian, 1974). Thus, it only rejects allocations that are unambiguously biased. Third, the allocations passing the test can be thought of as being the result of an eq-

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\(^6\)We assume budget balance, a realistic assumption in some environments, which simplifies our presentation. All of our results generalize if we assume free disposal of money, which may be relevant in some environments with negative externalities in which a Pareto improvement may be attained by burning money.

\(^7\)A sequence of allocations \( \{z^k\}_{k \in \mathbb{N}} \) converges as \( k \rightarrow \infty \) to \( z \equiv (x_i, \mu_i)_{i \in N} \) if it converges component-wise, i.e., for each \( i \in N \), \( x^k_i \rightarrow x_i \) and there is some \( K \in \mathbb{N} \) such that for each \( k \geq K \), \( \mu^k_i = \mu_i \).

\(^8\)A preference is continuous if its weak upper and weak lower contour sets are closed in \( Z \).
uitable process that gives equal opportunity to the agents (Kolm, 1971). Indeed, for an allocation \( z \), let \( B \) be the set of bundles that are allotted to the agents at \( z \). Imagine that one asks each agent to select out of the set \( B \) her preferred consumption bundles given that the final consumptions in the economy will be those in \( B \). This is a meaningful question for the agents in our environment. With anonymous externalities, fixing the final consumptions in the economy, the allocations that give each agent the same consumption bundle are welfare equivalent. If \( z \) is non-contestable, each agent selects a set of bundles containing her consumption at \( z \). Thus, one can think of \( z \) as the outcome of a process in which each agent is provided with the same opportunity set, \( B \), from which she selects her preferred choice.\(^9\)

There are alternative extensions of the no-envy test to economies with externalities (see Thomson, 2010). The most relevant to us is a proposal by Kolm (1995) that requires that no agent prefer the allocation at which all agents receive the consumption of some other agent, to the allocation at which all agents receive her own consumption, i.e., \( z \equiv (z_1, \ldots, z_n) \) is equitable if for each pair of agents \( i \) and \( j \), \((z_i, \ldots, z_i) R_i(z_j, \ldots, z_j) \). In our model, this requires rankings of allocations outside the space in which preferences are defined. Moreover, even if one extends preferences in an appropriate way, the test does not implement equal opportunity. Thus, we favor Foley’s original formulation.

An alternative approach to identify equitable allocations is to consider an intuitively equitable institution and the outcomes it produces. The most prominent such institution is an equal income market. Varian (1976) argues in favor of these allocations because they provide equal opportunity in a common set that is robust to changes in tastes and population. In our environment, the set of equal income competitive allocations and the set of non-contestable allocations coincide (see Sec. 5). Thus, by studying the existence and welfare properties of non-contestable allocations, we are also studying the existence and welfare properties of equal income competitive allocations.

Non-contestable allocations are desirable in environments beyond those with collective and symmetric ownership. Consider for instance a manager who has to allocate tasks and compensation among employees of the same rank. First, a non-contestable allocation promotes harmony in the workplace, for no employee finds the allocation is unambiguously biased towards another employee. Second, since they implement equal opportunity, they allow the manager to document non-discrimination in the workplace. This reduces the likelihood of a law suit.

Even without externalities, it is evident that non-contestable allocations may not exist if the budget is not enough to compensate agents for not receiving one object or the other. For instance, suppose that at least two agents care only about the object

\(^9\)We are abstracting from incentives issues, which can be analyzed only after allocations satisfying our normative requirements are known to exist.
they receive and prefer the same object. No matter how the budget is distributed, at least one of these agents will prefer to swap her consumption with the agent who receives her preferred object. Thus, in order to guarantee existence of non-contestable allocations one has to assume that some agents care about money and that the budget is enough to compensate them for not receiving one object or the other.

**Definition:** $u_i \in \mathcal{U}$ satisfies the **compensation assumption** if for each $z \in \mathbb{Z}_M^+$ at which agent $i$ receives a bundle with zero money, she weakly prefers to swap her consumption with some other agent who receives a positive consumption of money.\(^{10}\)

The compensation assumption is a statement about the budget in relation with the agent’s preferences. It requires that if the agent receives no money at an allocation, no matter how the budget is distributed among the other agents, she finds attractive to swap her allotment with at least one agent who receives some money. The assumption also implicitly bounds the agent’s externalities. For other-regarding preferences the requirement may not be satisfied for arbitrary large $M$ even if the utility of the agent is large when, ceteris paribus, her consumption of money is large. Thus, for it to be satisfied, it is also necessary that, ceteris paribus, the direct effect of a large consumption of money by the agent, dominate the externality on the other bundles (see our discussion of linear preferences leading up to Theorem 4).

We say that there are non-contestable allocations for $u \in \mathcal{U}^n$ when the budget is large enough if the following property is satisfied: there is $M_u$ such that if $M \geq M_u$, there are non-contestable allocations for $u$ in $\mathbb{Z}_M^+$.

### 3 Existence

Non-contestable allocations exist when agents care about money and there is enough money to distribute.

**Theorem 1.** Let $u \in \mathcal{U}^n$. If the utility functions of, at least, $n - 1$ agents satisfy the compensation assumption, there are non-contestable allocations for $u$ in $\mathbb{Z}_M^+$.

Theorem 1 proves possible our equity requirement in a model with a more realistic description of the agents than the self-regarding preferences model. Since the result is stated for an abstract domain, it is not immediate to realize what are its implications for preference domains of interest. In order to do so, we show in Sec. 4 that our results apply for inequity-averse preferences (Fehr and Schmidt, 1999) and for a general domain with linear externalities that we propose. We identify conditions on preferences in these domains guaranteeing they satisfy the compensation assumption when the budget is large enough.

\(^{10}\)Formally, for each $z \in \mathbb{Z}_M^+$ such that $x_i = 0$, there is a permutation $\pi$ that coincides with the identity function except for agents $\{i, j\}$ and such that $u_i(z_{\pi}) \geq u_i(z)$ and $x_{\pi(i)} > 0$. 
The proof of Theorem 1 is based on a combinatorial approach to equitable land division introduced by Stromquist (1980) and Woodall (1980) and refined by Su (1999). Part of our contribution is to show how these problems and our model with other-regarding preferences are connected. Land division is concerned with the equitable allocation of a measurable space among some agents whose preferences are represented by non-atomic measures. The first results in this literature provided existence of allocations at which each agent finds her assignment at least as good as that of the other agents (Dubins and Spanier, 1961). These results were unsatisfactory because assigned lots were unrestricted measurable sets, leaving the possibility that an agent would receive a countable union of small pieces of land. To solve this, the problem was redefined as the division of an interval, say [0, 1], into n subintervals among n agents. This provided an equitable division of any bounded set in a Euclidean space by means of n−1 “cuts.” This problem is equivalent to finding l in \( \Delta^{n-1} \), the standard simplex, at which each agent prefers a different subinterval in \{[0, l_1], [l_1, l_1 + l_2], ..., [l_1 + ... + l_{n-1}, 1]\}. This is relevant for our model with externalities. The utility of an agent from the \( k \)-th interval in \( l \in \Delta^{n-1} \) depends on \( l_1, ..., l_k \), for the location of the interval can only be determined when the length of the previous intervals is known. Our proof, which is based on the direct application of Sperner’s lemma introduced by Su (1999) to land division and chore allocation, reveals that the simplicial subdivision methods developed to solve this problem are suitable to handle anonymous consumption externalities in our model. Theorem 1, even when it is restricted to externality-free economies, is not a consequence of the previous results in this literature, however. In particular, it guarantees existence of non-contestable allocations for economies in which the preferences of one agent are not required to satisfy the compensation assumption.11 Using the covering method developed by Woodall (1980) one can prove our theorem under the additional assumption that each agent is indifferent to swap any two consumption bundles that have no money. This is very restrictive in our environment. In a contemporary paper Azrieli and Shmaya (2014) generalize the simplicial covering methods in Stromquist (1980) and Woodall (1980) to accommodate problems where an object may be shared by several agents. Again, our results are independent. Their covering methods require boundedness assumptions for all agents, so our theorem is not a consequence of theirs.

If externalities are non-anonymous, the conclusion of Theorem 1 may not hold.

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11Our theorem states that the requirement of the compensation assumption can be waived for one agent. Intuitively, if the budget is not enough to compensate one agent for not receiving a certain object, that agent can be allocated that object; as long as one can compensate all other agents, it is always possible to find a non-contestable allocation (note that this does not say that one can construct a non-contestable allocation without knowing the preferences of one agent). Our compensation assumption requirement cannot be dropped for more than one agent. That is, for a given \( M \) and preferences of \( n - 2 \) agents, one can easily find preferences for two additional agents such that the budget necessary for the existence of non-contestable allocations is greater than \( M \).
The issue is that non-anonymous externalities may limit the extent to which money can reduce the desire of an agent to swap her consumption with another agent.

**Example 1.** *(non-anonymous externalities)* Let \( A \equiv \{ \alpha, \beta, \gamma \} \) and \( N \equiv \{ 1, 2, 3 \} \). Agent 3 cares only about money, i.e., \( u_3(z) \equiv x_3 \). Preferences of the other agents are given by:

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\begin{align*}
    u_1(z) &\equiv \begin{cases} x_1 - 2 & \text{if } \mu_2 = \gamma \\ x_1 & \text{otherwise,} \end{cases} \\
    u_2(z) &\equiv \begin{cases} x_2 + 1 & \text{if } \mu_2 = \gamma \\ x_2 & \text{otherwise.} \end{cases}
\end{align*}
\]

These preferences are continuous and satisfy the compensation assumption when \( M \geq 2 \). Preferences of agents 2 and 3 are externality-free. In fact, they are quasi-linear. Agent 1’s preferences are non-anonymous, for her welfare depends on the object received by agent 2. There is no non-contestable allocation for this economy.

To see this, suppose that \( z \equiv (x, \mu) \) is non-contestable. Clearly, \( x_3 \geq \max\{x_1, x_2\} \).

Now, since agent 1 does not prefer to swap her consumption with agent 3, \( x_1 \geq x_3 \).

Since \( x_3 \geq x_1 \), we have that \( x_1 = x_3 \geq x_2 \). Thus, \( \mu_2 = \gamma \), for otherwise agent 2 would prefer to swap her consumption with the agent who receives object \( \gamma \). Consequently, \( x_2 \geq x_1 - 1 \), for otherwise agent 2 would prefer to swap her consumption with agent 1.

This implies that agent 1 prefers to swap her consumption with agent 2. Thus, \( z \) cannot be non-contestable for \( u \).

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### 4 Welfare analysis and economic domains

Non-contestability may be incompatible with efficiency. The issue here is that with other-regarding preferences, the utility of each member of a group of agents can increase by changing the profile of consumptions of money among them, while their aggregate consumption of money decreases. Thus, even in an economy in which ceteris paribus each agent prefers more money and for which efficient allocations exist, each non-contestable allocation may be Pareto dominated by extracting money from some agents and giving it to the rest (Example 2 in Appendix; see also our discussion leading up to Theorems 3 and 4).

This analysis suggests that the incompatibility of non-contestability and efficiency is the result of an aggregate imbalance between private and social interests. Indeed, when there are no externalities, non-contestable allocations of objects and money are always efficient (Svensson, 1983). Thus, one can expect that for a reasonable range of externalities there is general compatibility of these two properties. We show that this is so in two economic domains that capture natural other-regarding preferences.

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12Observe that our argument for non-existence of non-contestable allocations is independent of budget \( M \) and whether consumptions of money are required to be non-negative or not.
4.1 Inequity aversion

An inequity averse agent not only cares about her own consumption, but also about the equitability of the profile of consumptions. The following preferences capture these other-regarding preferences when each agent has equality of values as her own standard of equity. For each $z \equiv (x, \mu) \in Z$,

$$ u_i(z) \equiv v_{\mu_i}(z_i) - \frac{a_i}{n-1} \sum_{j \in N} \max\{v_{\mu_i}(z_j) - v_{\mu_i}(z_i), 0\} - \frac{c_i}{n-1} \sum_{j \in N} \max\{v_{\mu_i}(z_i) - v_{\mu_i}(z_j), 0\}, $$

where $a_i$ and $c_i$ are non-negative constants such that $c_i < 1$ and $c_i \leq a_i$; and for each $i$ and each $\alpha \in A$, $v^i_\alpha : \mathbb{R} \times A \to \mathbb{R}$ is a continuous, unbounded function, that is strictly monotone with respect to the first component, and represents the value that agent $i$, assigns to the consumption bundles when receiving object $\alpha$.\(^{13}\)

This domain of preferences, which we denote by $\mathcal{F}$, generalizes a proposal by Fehr and Schmidt (1999) for the allocation of money.\(^{14}\) Essentially, the agent takes her consumption as a reference to evaluate deviations from equality. Her ideal would be equality of values. The agent loses some welfare when other agents receive different consumption value than her. Ceteris paribus, she loses more welfare when the deviation is against her, than when the deviation is against others ($c_i \leq a_i$). The requirement $c_i < 1$ guarantees that, ceteris paribus, the agent prefers more money.

The value that an inequity-averse agent assigns to the consumption of another agent may depend on the object she receives. This is realistic in environments in which receiving a certain object may give the agent a particular vantage point that affects her assessment of others’ consumptions. In particular, this can reflect the “endowment effect,” i.e., an agent’s propensity to assign a different value to a bundle of goods when it is assigned to her, compared to when it is assigned to others (Kahneman et al., 1990). It turns out that, when this effect has a limit, non-contestability is compatible with the agents’ subjective assessment of equity. We say that the endowment effect is bounded for agent $i$ if there is $K > 0$ such that for each pair of objects $\alpha$ and $\beta$ and each $x \in \mathbb{R}$, $|v^i_\alpha(x, \alpha) - v^i_\beta(x, \alpha)| \leq K$.

Theorem 2. Let $u \in \mathcal{F}^n$ and suppose the endowment effect is bounded for, at least, $n-1$ agents. If the budget is large enough, there are non-contestable allocations for $u$.

The domain of quasi-linear inequity-averse preferences illustrates well the implications of Theorem 2. We say that an inequity averse preference, $u_i \in \mathcal{F}$, is quasi-linear when each $v^i_\alpha$ has the form $v^i_\alpha(x_\beta, \beta) \equiv v^i_\beta(\beta) + x_\beta$; here, $v^i_\beta(\beta)$ is the monetary value that agent $i$ assigns to receiving object $\beta$ when she receives object $\alpha$. Thus, the

\(^{13}\)Note that the functions $v^i_\alpha$ are anonymous, i.e., do not depend on the identities of the other agents, only on their consumption.

\(^{14}\)Characterizations of Fehr-Schmidt preferences are discussed by Neilson (2006); Sandbu (2008); Rohde (2010).
difference in the value that agent $i$ assigns to consumption bundle $(x_{\alpha}, \alpha)$ when she receives objects $\alpha$ and $\beta$ is just the difference in the monetary value assigned to $\alpha$ in both situations, i.e., $v_\alpha(\alpha) - v_\beta(\alpha)$. An immediate consequence of Theorem 2 is that non-contestable allocations exist when inequity-averse preferences are quasi-linear.

**Corollary 1.** Let $u \in \mathcal{F}^n$ be quasi-linear. If the budget is large enough, there are non-contestable allocations for $u$.

Non-contestable allocations for inequity-averse preferences may not be efficient when the equity concerns of the agents overwhelm their private interests. For this to be so, it must be the case that for some allocations it is possible to achieve a Pareto improvement by reassigning consumptions of money without reshuffling objects.

To see this, let $u \in \mathcal{F}^n$, $z \equiv (x, \mu)$ a non-contestable allocation for $u$, and $z' \in Z_M$. Let $z_\pi$ be the allocation obtained from $z$ by permuting consumption bundles so each agent receives the object she receives at $z'$. Since $z$ is non-contestable for $u$, no agent prefers $z_\pi$ to $z$. Thus, $z'$ can Pareto dominate $z$ only if it Pareto dominates $z_\pi$.

There are essentially two worst-case scenarios in which an allocation can be Pareto dominated for $u \in \mathcal{F}^n$ by reassigning money. Suppose that the consumption value of each agent $i \neq n$ is greater than her perceived consumption value for the consumption of agent $n$. If each of the first $n - 1$ agents give one unit of money to agent $n$, there can be a Pareto improvement if for each of these agents the reduction in inequality of value induced by the change offsets its direct effect. The following assumption precludes this is so.

**Assumption F1:** $u_i \in \mathcal{F}$ is such that there are $\overline{m}_i > \underline{m}_i$ such that for each pair of objects $\alpha$ and $\beta$, the derivative of $v^{\beta}_i(\cdot, \alpha)$ is bounded below by $\underline{m}_i$ and bounded above by $\overline{m}_i$ and,\(^{15,16}\)

$$\frac{c_i}{n-1} (n-1)\overline{m}_i + \underline{m}_i < \overline{m}_i.$$

Now, suppose that the consumption value of each agent $i \neq n$ is lower than her perceived consumption value for the allotment of each agent $j \in N \setminus \{i, n\}$, and is greater than her perceived consumption value for the allotment of agent $n$. If each agent $i \neq n$ gives one dollar to agent $n$, a Pareto improvement can be achieved not only because the inequality against agent $n$ is reduced, but also because the inequality against agent $i$ may be reduced. There are three ways in which one can guarantee this situation cannot compromise efficiency of a non-contestable allocation. The first is to require that it does not happen at all.

\(^{15}\)A similar observation was made by Dufwenberg et al. (2011) for marginal effects of income in indirect internal utility functions for inequity-averse agents in a general equilibrium setting.

\(^{16}\)We are implicitly assuming that the functions $v^{\beta}_i(\cdot, \alpha)$ are differentiable. This is only for simplicity in the presentation. The assumption can be replaced by the existence of $\overline{m}_i$ and $\underline{m}_i$ such that for each $x \in \mathbb{R}$ and each $\delta > 0$, $m_j \leq \frac{v^{\beta}_i(x+\delta, \alpha) - v^{\beta}_i(x, \alpha)}{\delta} \leq \overline{m}_i$. 


**Assumption F2**: \( u_i \in \mathcal{F} \) is such that there are bounds on marginal value of money \( m_i \) and \( \bar{m}_i \) as in F1 and

\[
\frac{c_i}{n-1} (m_i - \bar{m}_i) + (n-2) \frac{a_i}{n-1} (\bar{m}_i - m_i) < m_i.
\]

Second, it is enough to assume that there is no endowment effect. This implies that no agent perceives inequity against herself at each non-contestable allocation. Let \( z \equiv (x, \mu) \) be non-contestable for \( u \), \( z' \in Z \), and \( z_\pi \) the allocation obtained from \( z \) by permuting consumption bundles so each agent receives the object she receives at \( z' \). Let \( i \) be the agent whose consumption of money is reduced the most from \( z_\pi \) to \( z' \). Suppose that agent \( i \) gains in utility by the reduction in inequity against her by reassigning money from \( z_\pi \) to \( z' \). If this is so and there is no endowment effect for agent \( i \), she must prefer \( z \) to \( z_\pi \), for otherwise she would not perceive inequity against her at \( z_\pi \). It turns out that agent \( i \)'s loss from \( z \) to \( z_\pi \), is never compensated by the gain from \( z_\pi \) to \( z' \).

**Assumption F3**: \( u_i \in \mathcal{F} \) is such that for each pair of objects \( \alpha \) and \( \beta \),

\[
v_i^\alpha(\cdot, \alpha) = v_i^\beta(\cdot, \alpha).
\]

Finally, observe that assumption F2 precludes worst-case scenarios that would not arise if the agent's marginal value of money of her own consumption is always greater than that of her perceived marginal value of money for the consumption of the other agents.

**Assumption F4**: \( u_i \in \mathcal{F} \) is such that for each pair of different objects \( \alpha \) and \( \beta \), the derivative of \( v_i^\alpha(\cdot, \alpha) \) is bounded below and this bound is an upper bound of the derivative of \( v_i^\beta(\cdot, \beta) \).

**Theorem 3**. Let \( u \in \mathcal{F}^n \). Suppose that for each \( u_i \) one of the following is satisfied: (i) assumptions F1 and F3; (ii) assumption F2; (iii) assumptions F1 and F4. Then each non-contestable allocation for \( u \), \( z \in Z_M \), is efficient for \( u \).

If our assumptions in Theorem 3 are not satisfied, the result may not hold. In the appendix we construct \( u \in \mathcal{F}^n \) for which non-contestable inefficient allocations exist, and such that each \( u_i \) satisfies assumption F1 and violates assumptions F2-F4 (Example 3). This example materializes the worst-case scenario described before assumption F2.

Non-contestability and efficiency are compatible when inequity-averse preferences satisfy the assumptions of Theorems 2 and 3. These theorems reveal that, provided the lower bound on marginal value of money is positive for each agent, this always holds for an open set of parameters \( a_i \) and \( c_i \) that depends on the bounds on the marginal value of money. This set grows monotonically and continuously when the bounds get closer. In the limit, where marginal value of money is always constant and equal for all value functions of each agent, there are no restrictions on
and for large $n$ almost all $c_i$'s are acceptable, i.e., $c_i < (n - 1)/n$. Quasi-linear inequity-averse preferences are the preferences that reach this maximal compatibility of non-contestability and efficiency.\footnote{Quasi-linear preferences satisfy assumptions $F1$, $F2$, and $F4$, and may violate $F3$.}

**Corollary 2.** Let $u \in \mathcal{F}^n$ be quasi-linear and such that for each $i \in N$, $c_i < (n - 1)/n$. Then, each non-contestable allocation for $u$, $z \in Z_M$, is efficient for $u$.

### 4.2 Linear externalities

We introduce a domain of preferences in which the welfare of an agent is a linear function of the consumption of the other. Consider two agents, say $i$ and $j$, who receive objects $\alpha$ and $\beta$, at some allocation, respectively. The key assumption here is that the effect that agent $j$’s consumption of money, $x_j$, has on agent $i$’s welfare is a constant proportion of $x_j$. We denote this proportion by $c_i(\alpha, \beta)$ —note that this proportion may depend on the identity of the agent whose welfare is affected, the object that this agent receives, and the object that the other agent receives, but not on her identity. In other words, this proportion is independent of the identity of agent $j$. We assume quasi-linearity with respect to the agent's own consumption of money, i.e., for each $z \equiv (x, \mu) \in Z$,

$$u_i(z) \equiv v^i(\mu) + x_i + \sum_{j \neq i} c_i(\mu_i, \mu_j)x_j,$$

where $c_i(\mu_i, \mu_j) \in [0, 1)$ is the fraction of each dollar assigned to agent $j$ at $z$ that agent $i$ can use for her own benefit when she receives object $\mu_i$ and $v^i : A \to \mathbb{R}$. We denote the domain of these preferences by $\mathcal{L}$.

Our linear externalities model is a natural representation when objects are interconnected. Think for instance of fields that have to be irrigated (money in our model represents any infinitely divisible homogeneous good). Some proportion of water allocated to field $\alpha$ will end up serving field $\beta$. This proportion may depend on the distance between the fields and can be asymmetric, for one field may be higher than the other. Our domain can even accommodate situations in which this proportion depends on the agent who is assigned field $\alpha$. This is a realistic assumption when agents may have differential access to the technology that allows them to use the resources allocated to others. This flexibility is also desirable in other environments. For instance, consider a group of $n$ senior partners in a consulting firm who are responsible for the completion of a project; there are $n$ tasks, each partner can assume responsibility for only one task, and the firm has allocated a budget $M$ to complete the project. Each partner will receive a task and a share of the budget in order to assemble a team, buy equipment and software, pay travel expenses, etc.
Partners and their respective teams do not work in isolation, for tasks are not completely independent. Software that team \( \alpha \) purchases may also be of use to team \( \beta \). A travel intensive task may have less spillovers on the tasks that require more computational power, however. Thus, how easy or difficult is to guarantee the completion of a task for a partner depends on her own competence and experience for the task, her assigned budget, the budget assigned to each other task, and the ability of the partner to use the possible spillovers from the other tasks.

The compensation assumption may not be satisfied for any budget \( M > 0 \) for some linear externalities profiles. To illustrate this, assume that agent \( i \)'s valuation of all objects are equal and normalized to zero. Suppose further that when agent \( i \) receives object \( \alpha \), the effect that the consumption of money of each other agent has on her welfare is a proportion greater than or equal to \( 1/(n-1) \). That is, \( c_i(\alpha, \cdot) \geq 1/(n-1) \). Then, her utility when receiving object \( \alpha \) and no money will be greater than or equal to \( M \). If for each \( \beta \neq \alpha, c_i(\beta, \cdot) = 0 \), her utility from receiving an object different from \( \alpha \) and a budget share less than the entire budget will be less than \( M \). Thus, no matter how large \( M \) is, there will be allocations at which she would not swap receiving \( \alpha \) and no money for any other bundle. Fortunately, when we rule out this example, we guarantee that not only non-contestable allocations exist, but also they are efficient. We say that \( c > 0 \) strictly bounds externality coefficients for agent \( i \), if for each pair \( \alpha \) and \( \beta \), \( c_i(\alpha, \beta) < c \).

**Theorem 4.** Let \( u \in {\mathcal L}^n \). Suppose that \( 1/(n-1) \) strictly bounds externality coefficients for all agents. If the budget is large enough, there are non-contestable allocations for \( u \). Moreover, each non-contestable allocation for \( u, z \in Z_M \), is efficient for \( u \).

It is worth noting that \( 1/(n-1) \) is the greatest uniform bound on externality coefficients guaranteeing that the direct effect of one dollar allocated to an agent is greater than the aggregate indirect effect on the other agents.

5 Discussion

5.1 Non-contestability and the market

An intuitive equitable institution is the market when each agent is given an equal share of the aggregate income —here we cannot endow agents with equal parts of the aggregate endowment, for some goods are indivisible. This institution operates as follows in our environment. There is a vector of prices \( p \in \mathbb{R}^A \), in terms of money, for the indivisible goods. At these prices, the aggregate income in the economy is \( I_p \equiv M + \sum_{\alpha \in A} p_\alpha \). Each agent receives an equal share of \( I_p \) and maximizes her utility given the prices and income. More precisely, each agent chooses her best bundle among the ones she can afford, given that the final consumptions in the economy will be the bundles she can afford. Since externalities are anonymous,
this is a well defined utility maximization. Finally, market clears, i.e., the allocation is feasible.

**Definition:** \((p, z) \in \mathbb{R}^A \times Z \) is an equal income competitive equilibrium for \(u \in \mathcal{U}^n\), if for each \(i \in N\),
1. \(z_i \in B(p) \equiv \left\{ \frac{I_p}{n} - p_\alpha, \alpha \in A \right\}\).
2. \(z \in \arg\max_{z' \in Z} \{ u(z') : z' \in Z \text{ and for each } j \in N, \ z'_j \in B(p) \} \).
3. \(z \in Z_M\).

Equal income competitive allocations are non-contestable. More interestingly, each non-contestable allocation is decentralizable in an equal income market.\(^{18}\) That is, for each such allocation, say \(z\), there is a price vector \(p\) such that \((p, z)\) is an equal income competitive equilibrium.\(^ {19}\)

Given the equivalence of non-contestable allocations and the equal income competitive allocations, it is evident that Theorem 1, and its applications, i.e., Theorems 2, 4, 5, and 6 (below), provide conditions under which equal income competitive allocations exist. Conversely, one can ask whether previous literature providing existence of competitive equilibrium with externalities imply Theorem 1. To the length of our knowledge the only two relevant papers here are Gale (1984) and Dufwenberg et al. (2011). None of these works imply Theorem 1. Gale (1984) shows existence of competitive equilibria in an environment with indivisible goods were demand correspondences, the primitive of his model, may depend on the vector of nominal prices. In order to guarantee existence of competitive equilibria, Gale assumes that if the price of an object is large enough, the object is undesirable compared to a free object. If this assumption is satisfied, one cannot guarantee, for any budget, that prices are bounded by income in an equal income market. Dufwenberg et al. (2011) study conditions under which the market behavior of an agent with other-regarding preferences is indistinguishable from an agent with self-regarding preferences. Under these conditions standard existence results apply. In contrast, all of our results apply to economies in which externalities may affect the behavior of agents in the market. That is, if bundles \((x_\alpha, \alpha)\) and \((x_\beta, \beta)\) are both affordable to an agent at prices \(p\) and \(p'\), an agent with preferences in \(\mathcal{U}\) may demand \((x_\alpha, \alpha)\) at prices \(p\) and \((x_\beta, \beta)\) at prices \(p'\), depending on the effect that the other consumptions have on the welfare of the agent.

Our results in Sec. 4 are also relevant for general equilibrium analysis. Example 2 in Appendix implies that there can be situations in which no equal income competitive equilibrium is efficient. This impossibility is not the end of the road, however. Sec. 4 uncovers natural domains of other-regarding preferences in which the First Welfare Theorem for equal income competitive allocations holds, i.e., domains in

\(^{18}\)The connection between non-contestability and equal income competitive allocations when there are indivisibilities and no externalities was first noticed by Svensson (1983).

\(^{19}\)Take \(p\) such that for each \(\alpha \in A\), \(p_\alpha = \max_{\delta \in A} x_\delta - x_\alpha\).
which each equal income competitive outcome is efficient. These are the first results
to uncover such conditions. The closest result in previous literature is Dufwenberg
et al. (2011, Theorem 5), which provides conditions for a classical inequity-averse
economy, akin to our Theorem 3, under which a competitive equilibrium cannot be
Pareto dominated by an income redistribution at market prices. Our results are in-
dependent. Unlike receiving more income in a classical economy, receiving more
money than the agent who previously received a certain object, does not guarantee
a welfare gain to an agent. This requires we develop alternative bounds for an agent’s
welfare loss when there is a reallocation of resources.

5.2 Negative consumption of money

Our results are relevant for the existence of non-contestable allocations when con-
sumptions of money can be negative.\(^{20}\) Our welfare analysis already applies to this
case. In some circumstances, agents may have initial individual budgets that allow
them to compensate the other agents in order to get one object or the other. For
instance, in the allocation of tasks and budget shares among senior partners in a
consulting firm (Sec. 4.2) a partner can accept to give part of her division’s budget to
other divisions that will perform more demanding tasks. One needs also to consider
negative consumptions of money when the budget is negative, as in the allocations
of the rooms and the division of the rent among roommates.

Theorem 1 implicitly provides conditions for the existence of non-contestable
allocations when consumptions of money can be negative. The direct implication of
this theorem is that for a fixed preference profile, if the budget is large enough, then
there are non-contestable allocations at which each agent gets a non-negative con-
sumption of money. However, our choice of zero as minimal consumption of money
here can be easily modified. Instead we can require consumptions of money to be at
least some lower bound \(b\) and modify the compensation assumption accordingly.

**Definition:** \(u_i \in \mathcal{U}\) satisfies the **compensation assumption at (consumption
reference) \(b\)** if for each \(z \in Z_M\) at which each agent’s consumption of money is at
least \(b\) and \(x_i = b\), agent \(i\) weakly prefers to swap her consumption with some other
agent who receives a consumption of money greater than \(b\).

For a fixed budget and a fixed preference profile \(u\), if the compensation assump-
tion is eventually satisfied for \(b\) low enough, there are non-contestable allocations
for \(u\).\(^{21}\) (We omit the proof of the following result, which is a straightforward modi-
fication of our proof of Theorem 1).

\(^{20}\)This problem has been of independent interest in the literature, e.g., Alkan et al. (1991).

\(^{21}\) Another variant of Theorem 1 can be easily obtained by considering different minimal consump-
tions of money for each object, as in the allocation of tasks and salary with task-dependent minimal
salary constraints.
**Theorem 5.** Let \( u \in \mathcal{U}^n \). If the utility functions of, at least, \( n-1 \) agents satisfy the compensation assumption at \( b \), then there is \((x, \mu) \in Z_M\) that is non-contestable for \( u \) and such that for each \( i \in N \), \( x_i \geq b \).

The same conditions that guarantee existence of non-contestable allocations when the budget is large enough in Theorems 2 and 4, guarantee that these allocations exist if consumptions of money can be negative.

**Theorem 6.** Let \( M \in \mathbb{R} \). There are non-contestable allocations in \( Z_M \) for each \( u \in \mathcal{U}^n \) satisfying any of the following:

1. \( u \in \mathcal{F}^n \) and the endowment effect is bounded for, at least, \( n-1 \) agents.
2. \( u \in \mathcal{L}^N \) and \( 1/(n-1) \) strictly bounds externality coefficients for, at least, \( n-1 \) agents.

5.3 Different number of agents and objects

One can easily extend Theorem 1 to an environment were the number of agents and objects may differ. Our welfare analysis is not easily extended when there are more objects than agents, however. If there are no consumption externalities and \(|A| > n\), one can prove the existence of non-contestable efficient allocations by introducing \(|A| - n\) agents who only care about money. If \( z \) is a non-contestable allocation for the extended economy, the restriction of \( z \) to \( N \) is efficient (Alkan et al., 1991). Moreover, one can guarantee that for an arbitrary budget \( M \) there will be an extended economy for which the aggregate consumption of the agents in \( N \) is \( M \) (Velez, 2011). A modified version of Example 1 shows that this approach fails in our environment with externalities. Consider the economy with objects \( \{a, \beta, \gamma\} \) and agents \( \{1, 2\} \) as in the example. Note that their preferences are anonymous (here we interpret agent 1’s preference as being: \( u_1(z) = x_1 \) if no other agent receives \( \gamma \) and \( u_1(z) = x_1 - 2 \) if another agent receives \( \gamma \)). However, the economy in which we introduce a fictitious agent is our example for which there are no non-contestable allocations.

It is an open question to obtain conditions under which there are non-contestable and efficient allocations when there are more objects than agents.

6 Appendix

**Proof of Theorem 1.** Let \( u \in \mathcal{U}^n \) be such that the utilities of agents 1, ..., \( n-1 \) satisfy the compensation assumption. We assume without loss of generality that \( M = 1 \). We will construct a sequence of allocations that converges and its limit is a non-contestable allocation. A non-negative distribution of money is a vector \( x \equiv (x_a)_{a \in A} \) such that \( x \geq 0 \) and \( \sum_{a \in A} x_a = 1 \). Note that a distribution of money is a vector indexed by objects. Moreover, the space of distributions of money is isomorphic to the standard simplex \( \Delta^{n-1} \) (one can think of the set of objects as being \( \{a_1, ..., a_n\} \) and
$x \in \Delta^{n-1}$ being the distribution of money in which for each $i = 1, \ldots, n$, $x_{a_i} = x_i$.

Thus, for simplicity we will refer to the simplex of distributions of money $\Delta^{n-1}$. We will say that allocation $z \equiv (y, \mu)$ has distribution of money $x \equiv (x_a)_{a \in A}$ whenever for each $i \in N$, her consumption of money at $z$, i.e., $y_i$ is equal to $x_{\mu_i}$. Observe that all the allocations with a given distribution of money $x$ and at which agent $i$ receives object $a$ are obtained by permuting the consumptions of money among $N \setminus \{i\}$. Thus, since we assume anonymity of externalities, the set of allocations with distribution $x$ at which agent $i$ receives a given object, is an indifference class for agent $i$.

To construct the elements of our sequence of allocations, we will make use of the so-called baricentric triangulations of $\Delta^{n-1}$ (see Vick, 1994, for details). A triangulation of $\Delta^{n-1}$ is a collection of simplices with the same dimension as $\Delta^{n-1}$ that cover $\Delta^{n-1}$, with the property that if two simplices intersect, it has to be on a common face (a face of a simplex is the convex hull of a subset of its extremes, the simplex itself being the largest face; a facet is a face generated by $n-1$ vertices). The first baricentric triangulation of $\Delta^{n-1}$, which we denote by $T_1$, is defined inductively over $n$ as follows. When $n = 1$, i.e, the simplex is a single point, $T$ is the trivial covering $\{\Delta^{n-1}\}$. Now, suppose that the baricentric triangulations of $\Delta^k$ have been defined for $k = 1, \ldots, n-2$. Then the baricentric triangulation of $\Delta^{n-1}$ is the collection of sets conv.hull($\Delta, b$) where $\Delta$ runs over all simplices in the first baricentric triangulations of the facets of $\Delta^{n-1}$ and $b$ is the baricenter of $\Delta^{n-1}$. Figure 1(a) shows the first baricentric triangulation when $n = 3$. For a triangulation $T$, let $V(T)$ be the set of all the vertices of simplices in $T$. Notice that the vertices of $T_1$ are the baricenters of all the faces of $\Delta^{n-1}$. Thus, one can partition these vertices in those that are the baricenters of the faces generated by one vertex of $\Delta^{n-1}$, the baricenters of the faces generated by two vertices of $\Delta^{n-1}$, and so on.

The subsequent baricentric triangulations of $\Delta^{n-1}$ are obtained recursively as follows. Suppose that $T_1, \ldots, T_k$ have been defined. Then, $T_{k+1}$ is the collection of
all the covering subsimplices of the first baricentric triangulation of all simplices in $T_k$ (this is, $T_k$ contains all first baricentric triangulations of the triangles in $T_{k-1}$). Figure 1(b) shows the second baricentric triangulation, i.e., $T_2$, when $n = 3$. If we denote $T_0 \equiv \{\Delta^{n-1}\}$, for each $k > 1$, $V(T_k)$ is the set of all baricenters of the faces of all subsimplices in $T_{k-1}$.

An agent-labeling of a triangulation $T$ of $\Delta^{n-1}$ is a function that assigns an agent to each vertex of $T$, i.e., $L : V(T) \rightarrow \{1, \ldots, n\}$. We define an agent-labeling $L_k$ for the $k$-th baricentric triangulation of $\Delta^{n-1}$ as follows: assign to agent 1 the vertices that are the baricenters of the faces generated by one vertex of some $\Delta \in T_{k-1}$; assign to agent 2 the vertices that are the baricenters of the faces generated by two vertices of some $\Delta \in T_{k-1}$; and so on. Since any two subsimplices in $T_{k-1}$ can intersect only in a common face and representation inside a simplex is unique, $L_k$ is well defined.

Notice that agent $n$ gets assigned only interior vertices. Moreover, each subsimplex in $T_k$ is fully labeled, i.e., each of its extremes is assigned to a different agent.

We construct the $k$-th element of our sequence of allocations. Recall that for a fixed $x \in \Delta^{n-1}$, the allocations with distribution of money $x$ at which an agent receives a given object are an indifference class for the agent. For $x \in V(T_k)$ ask agent $L_k(x)$: *What are your preferred indifference classes from those associated with the objects when the distribution of money is $x$?* The answer to these questions can be recorded as the objects that are received by the agent in the maximal indifference classes. We define an object-labeling, i.e., a function $O_k : V(T_k) \rightarrow A$, that selects for each $x \in V(T_k)$, one of the answers of agent $L_k(x)$, breaking ties in favor of objects with positive amount of money at $x$. Since the utility of agents $1, \ldots, n - 1$ satisfy the compensation assumption, and agent $n$ only $L_k$-labels interior points of $\Delta^{n-1}$, we are able to select as answer an object, say $\alpha$, that receives a positive amount of money at $x$, i.e., $x_\alpha > 0$. Thus, the object-labeling $O_k$ is a Sperner’s labeling, i.e., the $O_k$-label of each vertex of $\Delta^{n-1}$ is different and for each $x \in V(T_k)$, $O_k(x)$ is one of the labels of the vertices of $\Delta^{n-1}$ that generate $x$. By a well known result in combinatorics, Sperner’s lemma, for each Sperner’s labeling of $T_k$, there is an odd number of fully labeled simplices in $T_k$ (see Border, 1999; McLennan and Tourky, 2008; Su, 1999, for the statement and elemental proofs of Sperner’s lemma). Thus, there is at least a subsimplex in $T_k$ that is fully $O_k$-labeled. We define the $k$-th element of the sequence of allocations as follows. Take $\Delta \in T_k$ that is fully $O_k$-labeled and let $x \in \Delta$. From the bundles $\{(x_\alpha, \alpha)\}_{\alpha \in A}$ assign to agent $i \in N$ the bundle that contains her answer in the question for the vertex of $\Delta$ whose $L_k$-label is $i$. Since each agent was assigned one of the extremes of $\Delta$ and $\Delta$ is fully $O_k$-labeled, this defines a feasible allocation, $z^k$.

Since there is a finite number of objects, we can select a subsequence of $\{z^k\}_{k \in \mathbb{N}}$ in which each agent receives the same object.\textsuperscript{22} Since $\Delta^{n-1}$ is a compact set, we can

\textsuperscript{22}One can construct this sequence by refining $\{z^k\}_{k \in \mathbb{N}}$ agent by agent. More precisely, since there is a finite number of objects, there has to be an object that agent 1 receives in infinitely many elements of the sequence. Thus, one can pass to a subsequence in which agent 1 receives the same object. From
select a subsequence whose distributions converge to a limit $x^*$. Thus, this subsequence of allocations converges to an allocation $z^*$ with distribution of money $x^*$ and in which each agent receives the object she receives in each element of the sequence. Since the mesh size of the baricentric triangulations vanish as $k \to \infty$ (Vick, 1994), for each agent and each neighborhood of the limit allocation, there is an allocation at which the agent receives the object in the limit allocation and at which the agent would not prefer to swap her consumption with any other agent. By continuity of preferences, no agent prefers to swap her consumption with any other agent and the limit allocation is non-contestable. \hfill \Box

**Example 2.** ($u \in \mathcal{U}^n$ satisfying the compensation assumption for each agent, such that the set of efficient allocations for $u$ is non-empty, but no non-contestable allocation for $u$ is efficient for $u$.) We assume that the budget is zero and drop the non-negativity requirement in consumptions of money; $u$ can be easily shifted to accommodate non-negative consumptions. $A \equiv \{a, \beta, \gamma\}$ and $N \equiv \{1, 2, 3\}$. For each $i \in N$ and each $z \equiv (x, \mu) \in Z_M$,

$$ u_i(z) \equiv \begin{cases} 
 x_i + f(x_{\beta}) & \text{if } \mu_i = a \\
 f(x_{\alpha}) + x_i & \text{if } \mu_i = \beta \\
x_i & \text{if } \mu_i = \gamma
\end{cases} \quad \text{where} \quad f(a) \equiv \begin{cases} 
 0 & \text{if } a \in ]-\infty, -2[ \\
d + 2 & \text{if } a \in ]-2, -1[ \\
 -a & \text{if } a \in ]-1, 0[ \\
 0 & \text{if } a \in [0, +\infty].
\end{cases}
$$

One can easily see that $u_i$ satisfies the compensation assumption for $M \geq 0$. Let $z \equiv (x, \mu) \in Z_M$ be the allocation $z_1 \equiv (-4, a)$, $z_2 \equiv (-4, \beta)$, and $z_3 \equiv (8, \gamma)$. We claim that $z$ is efficient for $u$. Let $z' = (x', \mu') \in Z_M$ be such that $z'Rz$. Then, $x'_3 \geq 8$. Feasibility implies that another agent gets at most $-4$. Suppose that $x'_2 \leq -4$. We claim that $x'_2 = -4$. Clearly if $\mu_2 = \gamma$, $x_2 = -4$. Suppose that $\mu_2 = a$. The only way that agent 2 can get a utility boost from the consumption of the agent who gets $\beta$ is that $x_\beta \in [-2, 0]$. But if $x_\beta \in [-2, 0]$, $x_2 \leq -6$ and $u_2(z') < -4$. Thus, $x_2 = -4$. The problem is symmetric when $\mu_2 = \beta$. The same argument shows that $x_1 = -4$. Then, $x' = x$. Thus, $zRz'$. Now, at each non-contestable allocation for $u$, consumptions of money have to be zero. If there is a consumption that is positive, the consumption of the agent who receives $\gamma$ has to be positive. The utility of the agent who receives the minimal consumption of money is at most zero. Thus, that agent will prefer to swap

---

\[\text{footnote}{\text{23Let } r > 0 \text{ and consider the open ball in } \Delta^{n-1} \text{ centered at } x^* \text{ with radius } r/2. \text{ For } k \text{ large enough, } x^k, \text{ i.e., the distribution of money in } z^k, \text{ is } r/2\text{-close to } x^* \text{ because as } k \to \infty, x^k \to x^*. \text{ Now, for } k \text{ large enough, the mesh size of } T_k, \text{ i.e., the maximum diameter of a subsimplex, is less than } r/2. \text{ Let } k \text{ be large enough so these two conditions are satisfied and } \Delta_k \text{ the subsimplex that defines } z^k. \text{ Consider an agent, say } i \in N. \text{ This agent labels one of the vertices of } \Delta_k \text{ in labeling } L_k. \text{ Let } v^{i,k} \text{ be this vertex. Suppose that the object labeling of this vertex is } a. \text{ That means that agent } i \text{ receives object } a \text{ both at } z^k \text{ and } z. \text{ Moreover, among all allocations with distribution of money } v^{i,k} \text{ agent } i\text{'s preferred allocations are those at which she receives } a. \text{ Since the diameter of } \Delta_k \text{ is at most } r/2, \text{ then } v^{i,k} \text{ is } r \text{ close to } x^*.}\]
her consumption with the agent who receives $\gamma$. If consumptions of money are all zero, there is a Pareto improvement by taking $\varepsilon \in (0,1)$ from each agent who receives $\alpha$ and $\beta$ and giving $2\varepsilon$ to the agent who receives $\gamma$. □

**Example 3.** ($u \in \mathcal{F}^n$ and $z \equiv (x,\mu) \in Z_M$ that is non-contestable for $u$ such that: $u$ satisfies $F1$; $u$ violates $F2$ with the minimal slack possible; $u$ also violates $F3$ and $F4$; the set of efficient allocations for $u$ is non-empty; and $z$ is not efficient for $u$.) Our example has $n \geq 3$ agents and the budget is $n-1$. For simplicity we label objects $A \equiv \{1,2,\ldots,n\}$. Let $z$ be the allocation at which agent $i$ receives object $i$; agents $1,\ldots,n-1$ receive $S$ each, and agent $n$ receives $S0$. We construct preferences such that the money derivative of $v^1_i(\cdot,\alpha)$ is bounded below by 1 and above by $K$, where $K \in (1,2)$.

We only make requirements for each $i$ on the functions $v^1_i(\cdot,j)$ with $j \in A$, i.e., the value function for agent $i$ when she receives object $i$. One can define $v^1_i(\cdot,\cdot)$ for $j \neq i$ such that the allocation is non-contestable for $u$. This requires we violate $F3$. For each agent $i = 1,\ldots,n-1$, $c_i \equiv 0$ and $a_i \equiv (n-1)/((n-2)(K-1))$. Thus, $F2$ will be violated with the minimal slack for these agents. For agent $n$, $c_n = a_n \equiv 0$. We specify preferences for agent $1$ such that if agent $1$ gives up $1/n$ in consumption of money, each agent from $\{2,\ldots,n-1\}$ does the same, and agent $n$ receives the $(n-1)/n$ collected from agents $1,\ldots,n-1$, agent $1$ is not worse off when there is no reshuffling of objects. By symmetry, one can specify preferences for agents $2,\ldots,n-1$ such that each of them is not worse off with the change. Since money is desirable for agent $n$, she is better off. Thus, our allocation is not efficient for $u$. Our preference for agent $1$ is such that for each $j \neq 1$, $v^1_i(1,1) = v^1_i(1-(1-1/K)/n,j) \equiv 0$, $v^1_i(1-1/n,1) = v^1_i(1-1/n,j) \equiv -1/n$, and $v^1_i(1,j) \equiv K(1-1/K)/n$. One can complete the definition of these functions such that the derivative of $v^1_i(\cdot,1)$ is 1 and the derivative of each $v^1_i(\cdot,j)$ with $j \neq 1$ is $K$ (since our example holds for arbitrary $K > 1$, one can say that $u$ violates $F4$ for an arbitrarily small margin). Now, agent $1$ is not worse off with the redistribution of money whenever

$$v^1_i(1-1/n,1) \geq v^1_i(1,1) - \frac{a_1}{n-1} \left( \sum_{j=2,\ldots,n-1} v^1_i(1,j) - v^1_i(1,1) \right) .$$

This inequality holds with equality when $a_1 \equiv (n-1)/((n-2)(K-1))$.

Finally, note that since consumption externalities are negative for each $u \in \mathcal{F}$, there is a level of utility $\overline{u}$ that cannot be surpassed by each agent simultaneously at an allocation in $z \in Z_M$. By continuity, the set $\arg\max_{z \in Z_M} \min_{i \in N} u_i(z)$ is non-
empty; each allocation that, for some order, lexicographically maximizes the utility levels of the agents in this set is efficient for \( u \). □

**Proof of Theorem 2.** Let \( K > 0 \) and \( u_i \in \mathcal{F} \) be such that for each pair \( \alpha \) and \( \beta \), and each \( x \in \mathbb{R} \), \( |v^i_\alpha(x, \beta) - v^i_\beta(x, \alpha)| \leq K \). We will prove a general result that will also help us proving Theorem 6. Let \( b \in \mathbb{R} \) be such that \( M > nb \) and suppose that for each pair \( \{\alpha, \beta\} \subseteq A \), \( v^i_\beta(M/n, \beta) - v^i_\alpha(b, \alpha) > K \). Let \( z \equiv (x, \mu) \in Z_M \) and suppose that \( x_i = b \) and for each \( j \neq i \), \( x_j \geq b \). Since \( \sum_{j \in N} x_j = M \), there is \( j \in N \) such that \( x_j \geq M/n > b \). Since value functions are increasing in money, then for each pair \( \{\alpha, \beta\} \subseteq A \), \( v^i_\beta(x, \beta) - v^i_\beta(b, \alpha) > K \). Let \( \tilde{z} \) be the allocation obtained by swapping the consumption bundles of agents \( i \) and \( j \) at \( z \). To simplify notation we denote \( \beta = \mu_j \) and \( \alpha = \mu_i \). Let \( B \equiv \{\gamma \in A : v^i_\beta(x, \gamma) > v^i_\beta(x, \gamma)\} \) and \( C \equiv \{\gamma \in A : v^i_\alpha(x, \alpha) > v^i_\alpha(x, \gamma)\} \). We obtain conditions guaranteeing \( u_i(\tilde{z}) > u_i(z) \), i.e.,

\[
v^i_\beta(x, \beta) - \frac{c_i}{n-1} \sum_{\gamma \in B} v^i_\beta(x, \beta) - v^i_\beta(x, \gamma) - \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} v^i_\beta(x, \beta) - v^i_\beta(x, \gamma) + v^i_\alpha(x, \gamma) > v^i_\alpha(b, \alpha).
\]

Let \( \gamma \in A \setminus B \). Thus, \( v^i_\alpha(x, \gamma) - v^i_\beta(x, \beta) \geq 0 \). Moreover, \( v^i_\alpha(x, \gamma) - v^i_\alpha(b, \alpha) \geq v^i_\beta(x, \gamma) - v^i_\beta(x, \beta) \), for otherwise \( v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) < v^i_\beta(x, \gamma) - v^i_\beta(x, \gamma) \leq K \). Thus, \( \gamma \in A \cap C \). Thus, \( A \cap B \subseteq A \setminus C \) and \( u_i(\tilde{z}) > u_i(z) \) whenever:

\[
v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) - \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} v^i_\beta(x, \beta) - v^i_\beta(x, \gamma) + v^i_\alpha(x, \gamma) + v^i_\alpha(x, \gamma) > 0.
\]

Since for each \( \gamma \in A \), \( v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) > K \geq v^i_\beta(x, \gamma) - v^i_\alpha(x, \gamma) \), this inequality is satisfied whenever:

\[
v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) - \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} v^i_\beta(x, \beta) - v^i_\beta(x, \gamma) + v^i_\alpha(x, \gamma) + v^i_\alpha(x, \gamma) > 0.
\]

For each \( \gamma \in A \setminus C \), \( v^i_\beta(b, \alpha) - K \leq v^i_\alpha(x, \gamma) - K \leq v^i_\beta(x, \gamma) \). Thus, the inequality above holds whenever:

\[
v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) - \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} v^i_\beta(x, \beta) - v^i_\beta(x, \gamma) + v^i_\alpha(x, \gamma) + v^i_\alpha(x, \gamma) > 0.
\]

Since, \( |B| \leq n-1 \), this inequality holds whenever:

\[
(1 - c_i)(v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha)) + \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} v^i_\beta(x, \gamma) - v^i_\alpha(x, \gamma) - \frac{c_i}{n-1} \sum_{\gamma \in B \cap C} K > 0.
\]

Since for each \( \gamma \in A \), \( v^i_\beta(x, \gamma) - v^i_\alpha(x, \gamma) \geq -K \), this inequality holds whenever, \( v^i_\beta(x, \beta) - v^i_\alpha(b, \alpha) > c_i K/(1 - c_i) \).
Thus, there are two cases. (\(v_{\alpha} > \beta\))

Case 1: \(\gamma \equiv \{x' \in \Gamma : x' \equiv \alpha \}\). Since \(\gamma \equiv \{x' \in \Gamma : x' \equiv \alpha \}\) is obtained from \(x\) by reshuffling consumption bundles, and for each \(i \in N\), \(u_i(z') \geq u_i(z)\), it must be the case that for each \(a \in A\), \(x'_a \geq x_a\). This implies that \(z'\) is obtained from \(z\) by reshuffling consumption bundles, and for each \(i \in N\), \(u_i(z) \geq u_i(z')\). Suppose by means of contradiction that there is \(a \in A\) such that \(x'_a < x_a\). Suppose without loss of generality that \(a \in \arg \max \gamma \in A \gamma \times y - x'\). Let \(j \in N\) be such that \(\mu'_j = a\).

Suppose first that \(u_j\) satisfies \(F1\) and \(F4\). Since \(z\) is non-contestable for \(u\), then \(u_j(z) \geq u_j(\hat{z})\) where \(\hat{z}\) is obtained from \(z\) by swapping the consumption bundles of agent \(j\) and the agent who receives \(a\) at \(z\). We prove that \(u_j(\hat{z}) > u_j(z)\), which implies \(u_j(\hat{z}) > u_j(z)\), a contradiction. In what follows, we will only analyze statements of preference for agent \(j\) when receiving object \(a\). In order to simplify notation, we drop the superindices and subindices associated with this agent and this object in \(v_{\alpha}(\cdot, \cdot), m_j\), and \(\tilde{m}_{ij}\). Let \(\Delta u \equiv u_j(\hat{z}) - u_j(z')\) and for each \(\gamma \in A\), \(\Delta v(\gamma) \equiv v(x, \gamma) - v(x', \gamma)\). For each \(y \in \mathbb{R}^A\) and each \(\gamma \in A\), let

\[
\Phi(y, \gamma) \equiv -a_j \max\{v(y, \gamma) - v(y, \gamma)\} - c_j \max\{v(y, \gamma) - v(y, \gamma)\}.
\]

Then,

\[
\Delta u = \Delta v(a) + \frac{1}{n-1} \sum_{\gamma \in A} \Phi(x, \gamma) - \Phi(x', \gamma).
\]

For each \(\gamma \in A\), let \(\Delta y \equiv \Phi(x, \gamma) - \Phi(x', \gamma)\). Let \(A_1 \equiv \{\gamma \in A : \nu(x, \gamma) \leq v(x', \gamma)\}\) and \(A_1 \equiv \{\gamma \in A : \nu(x', \gamma) \leq v(x, \gamma)\}\) and \(\gamma \in A_1\). Note \(A_1 \neq \emptyset\), for \(a \in A_1\). We claim that \(\Delta y \geq -c_j(\Delta v(a) - \Delta v(\gamma))\). There are two cases.

Case 1: \(v(x', \gamma) \leq v(x', \gamma)\). Then, \(\Delta y = -c_j(v(x, \gamma) - v(x', \gamma)) + c_j(v(x', \gamma) - v(x', \gamma))\).

Case 2: \(v(x', \gamma) > v(x', \gamma)\). Then, \(\Delta y = -c_j(v(x, \gamma) - v(x', \gamma)) + a_j(v(x', \gamma) - v(x', \gamma)) \geq -c_j(v(x, \gamma) - v(x', \gamma)).\) Now, \(v(x, \gamma) - v(x', \gamma) = v(x, \gamma) - v(x', \gamma) + v(x, \gamma) - v(x', \gamma)\) By the case hypothesis, \(v(x, \gamma) > v(x', \gamma)\). Thus, \(v(x, \gamma) - v(x', \gamma) < v(x, \gamma) - v(x', \gamma) + v(x', \gamma) - v(x', \gamma)\). Thus, \(\Delta y > -c_j(v(x, \gamma) - v(x', \gamma) + v(x', \gamma) - v(x', \gamma))\).

We claim that for each \(\gamma \in A \setminus A_1\), \(\Delta y \geq 0\). Let \(\gamma \in A \setminus A_1\). Then, \(v(x, \gamma) > v(x, \gamma)\). There are two cases.

Case 1: \(x' \geq x\). Then, \(v(x', \gamma) \geq v(x, \gamma)\). Since \(\gamma \in A \setminus A_1\), \(v(x, \gamma) > v(x, \gamma)\). Thus, \(v(x', \gamma) > v(x', \gamma)\). Thus, \(\Delta y = -a_j(v(x, \gamma) - v(x, \gamma) + a_j(v(x', \gamma) - v(x', \gamma)) = a_j(v(x, \gamma) - v(x, \gamma) + v(x', \gamma) - v(x, \gamma) \geq 0\).

Case 2: \(x' < x\). By \(F4\), the derivative of \(v(\cdot, \gamma)\) is always greater than or equal to that of \(v(\cdot, \gamma)\). Since \(x, x' \geq x, x' \geq x, v(x', \gamma) - v(x', \gamma) \leq v(x, \gamma) - v(x, \gamma)\). Thus,
Thus, \( v(x, \gamma) - v(x_a, \alpha) \leq v(x', \gamma) - v(x', \alpha) \). Since \( \gamma \in A \setminus A_1 \), then \( 0 < v(x, \gamma) - v(x_a, \alpha) \). Thus, \( v(x', \alpha) < v(x', \gamma) \). Thus, \( \Delta \gamma = -a_j(v(x, \gamma) - v(x_a, \alpha)) + a_j(v(x', \gamma) - v(x', \alpha)) = a_j(v(x_a, \alpha) - v(x_a, \alpha) - [v(x, \gamma) - v(x', \gamma)]) \). Recall that \( v(x, \gamma) - v(x', \gamma) \leq v(x_a, \alpha) - v(x', \alpha) \). Thus, \( \Delta \gamma \geq 0 \).

Since for each \( \gamma \in A_1 \), \( \Delta \gamma \geq -c_j(\Delta v(a) - \Delta v(\gamma)) \) and for each \( \gamma \in A \setminus A_1 \), \( \Delta \gamma \geq 0 \),

\[
\Delta u \geq \Delta v(a) - \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(a) + \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(\gamma). \tag{1}
\]

Let \( B \equiv \{ \gamma \in A_1 : x_{\gamma} \leq x_{\gamma}' \} \). Recall that by \( F1 \), the derivative of \( v(\cdot, \gamma) \) is bounded above by \( \bar{m} \) and below by \( \bar{m}' \). Thus, if \( \gamma \in B \), then \( x_{\gamma}' \geq x_{\gamma} \) and \( \Delta v(\gamma) \geq \bar{m}(x_{\gamma} - x_{\gamma}') \); if \( \gamma \in A_1 \setminus B \), then \( x_{\gamma}' < x_{\gamma} \) and \( \Delta v(\gamma) \geq m(x_{\gamma} - x_{\gamma}') \). Thus, \( \sum_{\gamma \in A_1} \Delta v(\gamma) \geq (\bar{m} - m) \sum_{\gamma \in B} (x_{\gamma} - x_{\gamma}') + m \sum_{\gamma \in A_1 \setminus B} (x_{\gamma} - x_{\gamma}') \).

We claim that for each \( C \subseteq A, \sum_{\gamma \in C} x_{\gamma} - x_{\gamma}' \geq \frac{(1/m)}{\bar{m}} \sum_{\gamma \in A \setminus C} \Delta v(a) \). Since \( z \in Z^+ \), then \( \sum_{\gamma \in A} x_{\gamma}' \leq \sum_{\gamma \in A} x_{\gamma} \).

Thus, \( \sum_{\gamma \in C} x_{\gamma} - x_{\gamma}' \geq \sum_{\gamma \in A \setminus C} x_{\gamma} - x_{\gamma}' \geq \sum_{\gamma \in A \setminus C} x_{\gamma} - x_{\gamma} \), where the last inequality holds because for each \( \gamma \in A \), \( x_a - x_a' \geq x_{\gamma} - x_{\gamma}' \). By \( F1 \), \( \Delta v(a) \geq m(x_a - x_a') \). Thus, \( x_{\gamma}' - x_a \geq \frac{1}{m} \Delta v(a) \) and \( \sum_{\gamma \in C} x_{\gamma} - x_{\gamma}' \geq \frac{1}{m} \sum_{\gamma \in A \setminus C} \Delta v(a) \).

Thus, \( \sum_{\gamma \in A} \Delta v(\gamma) \geq \frac{(\bar{m} - m)}{m} \sum_{\gamma \in A} \Delta v(a) - \sum_{\gamma \in A \setminus A_1} \Delta v(a) \). Combining this inequality with inequality (1), we have that:

\[
\Delta u \geq \Delta v(a) \left( 1 - \frac{c_j}{n-1} \left( |A_1| + |A \setminus A_1| + \left( \frac{\bar{m} - m}{m} \right) 1_{B \neq \emptyset} |A \setminus B| \right) \right),
\]

where \( 1_{B \neq \emptyset} \) is the indicator function that equal to one when \( B \neq \emptyset \) and zero otherwise. Thus,

\[
\Delta u \geq \Delta v(a) \left( 1 - \frac{c_j}{n-1} \left( n + \left( \frac{\bar{m} - m}{m} \right) (n-1) \right) \right) = \Delta v(a) \left( 1 - c_j \left( \frac{1}{n-1} + \frac{m}{\bar{m}} \right) \right).
\]

By assumption \( F1 \), \( \Delta u > 0 \).

Now, suppose that \( u \) satisfies \( F2 \) (\( F2 \) implies \( F1 \)). Note that the only place that \( F4 \) was used above was to prove that for each \( \gamma \in A \setminus A_1 \) such that \( x_{\gamma}' < x_{\gamma} \), \( \Delta \gamma \geq 0 \). We prove now that for each such \( \gamma \), \( \Delta \gamma \geq -a_j \left( \frac{m}{m} \right) \Delta v(a) \). Since \( \gamma \in A \setminus A_1 \),

\[
\Delta \gamma = -a_j (v(x, \gamma) - v(x_a, \alpha)) + c_j \max \{ v(x_a', \alpha) - v(x', \gamma), 0 \}
+ a_j \max \{ v(x', \gamma) - v(x_a', \alpha), 0 \}.
\]

Since \( x_{\gamma} > x_{\gamma}' \) and \( v(x, \gamma) - v(x', \gamma) \leq \frac{m}{m}(x_{\gamma} - x_{\gamma}') \leq m(x_a - x_a') \leq \frac{m}{m} (v(x_a, \alpha) - v(x_a', \alpha)) \). Then,

\[
v(x, \gamma) - v(x', \gamma) \leq \frac{m}{m} \Delta v(a). \tag{2}
\]

\footnote{Observe that our argument applies when there is free disposal of money, for a modified bound in which the set \( D \) at the end of our estimation can be taken to be such that \( |D| = n - 1 \).}
There are two cases.

**Case 1:** \(v(x'_j, y) < v(x'_a, a)\). Then, \(\Delta_y = -a_j(v(x'_j, y) - v(x'_a, a)) + c_j(v(x'_a, a) - v(x'_a, y))\). By the case hypothesis, \(v(x'_a, a) > v(x'_a, y)\). Thus, \(\Delta_y \geq -a_j(v(x'_j, y) - v(x'_a, a))\). Now, \(v(x'_j, y) - v(x'_a, a) = v(x'_j, y) - v(x'_j, y) + v(x'_j, y) - v(x'_a, a) + v(x'_a, a) - v(x'_a, a)\). By the case hypothesis, \(0 > v(x'_j, y) - v(x'_a, a)\). Thus,

\[
v(x'_j, y) - v(x'_a, a) \leq v(x'_j, y) - v(x'_a, y) - (v(x'_a, a) - v(x'_a, a)) \leq \left(\frac{n}{n-1} - 1\right) \Delta v(a),
\]

where the last inequality follows from equation (2).

**Case 2:** \(v(x'_j, y) \geq v(x'_a, a)\). Then, \(\Delta_y = -a_j(v(x'_j, y) - v(x'_a, a)) + a_j(v(x'_a, y) - v(x'_a, a))\). Thus, \(\Delta_y = -a_j(v(x'_j, y) - v(x'_a, y)) + a_j(v(x'_a, a) - v(x'_a, a))\). Our claim then follows from equation (2).

Let \(D \equiv \{y \in A \setminus A_1 : x'_j < x'_a\}\). Then,

\[
\Delta u \geq \Delta v(a) - \frac{c_j}{n-1} \sum_{y \in A_1} \Delta v(a) + \frac{c_j}{n-1} \sum_{y \in A_1} \Delta v(y) - \frac{a_j}{n-1} \sum_{y \in D} \left(\frac{n}{n-1} - 1\right) \Delta v(a).
\]

The remaining of the argument when \(F1\) and \(F4\) hold, is valid. Thus,

\[
\Delta u \geq \Delta v(a) \left(1 - \frac{c_j}{n-1} \left(\frac{1}{n-1} + \frac{n}{n-1}\right) - \frac{a_j}{n-1} \left(\frac{n}{n-1} - 1\right) |D|\right).
\]

The cardinality of \(D\) is at most \(n - 2\), for otherwise \(\sum_{y \in A} x'_j < \sum_{y \in A} x'_a\). Thus,

\[
\Delta u \geq \Delta v(a) \left(1 - c_j \left(\frac{1}{n-1} + \frac{n}{n-1}\right) - a_j \frac{n-2}{n-1} \left(\frac{n}{n-1} - 1\right)\right).
\]

By \(F2\), \(\Delta u > 0\).

Finally, suppose that \(u_j\) satisfies \(F1\) and \(F3\). Since \(F3\) holds, we continue dropping the subindex \(a\) in \(v'_a(\cdot, \cdot)\). Since we will only make statements about the rankings of agent \(j\), we will also continue dropping the subindex and superindex \(j\) from the notation.

Let \(s \equiv (y, \sigma) \in Z\) and \(s'\) be the allocation obtained from \(s\) by swapping the consumption bundles of agents \(j\) and \(i\). We claim that \(u_j(s) \geq u_j(s')\) if and only if \(v(s_j) \geq v(s_i)\). By definition of \(u_j\), if \(v(s_j) = v(s_i)\), then \(u_j(s) = u_j(s')\). We prove that if \(v(s_j) > v(s_i)\), then \(u_j(s) > u_j(s')\), which completes the proof of our claim. Denote \(\sigma_j \equiv \delta\) and \(\sigma_i \equiv \eta\). Let \(A_1 \equiv \{y \in A : v(y, \gamma) \geq v(y, \delta)\}\) and \(B_1 \equiv \{y \in A : v(y, \gamma) \geq v(y, \eta)\}\). Then, \(u_j(s) > u_j(s')\) whenever

\[
\begin{align*}
&v(y, \delta) - \frac{c_j}{n-1} \sum_{y \in A \setminus A_1} v(y, \delta) - v(y, \gamma) - \frac{a_j}{n-1} \sum_{y \in A \setminus B_1} v(y, \gamma) - v(y, \delta) > \\
v(y, \eta) - \frac{c_j}{n-1} \sum_{y \in A \setminus B_1} v(y, \eta) - v(y, \gamma) - \frac{a_j}{n-1} \sum_{y \in B_1} v(y, \gamma) - v(y, \eta).
\end{align*}
\]

\(\text{If we assume free disposal of money, \(F2\) must be modified to reflect the possibility that } |D| = n - 1.\)
Since \( v(y_\delta, \delta) > v(y_\gamma, \eta) \), then \( A_1 \subseteq B_1 \) and \( A \setminus B_1 \subseteq A \setminus A_1 \). Thus, the inequality above is satisfied whenever
\[
\frac{c_j}{n-1} \sum_{\gamma \in (A \setminus A_1) \setminus (A \setminus B_1)} v(y_\delta, \delta) - v(y_\gamma, \gamma) > 0.
\]
Since for each \( \gamma \in (A \setminus A_1) \setminus (A \setminus B_1) \), \( v(y_\delta, \delta) \leq v(y_\gamma, \gamma) \), the inequality above is satisfied whenever \( v(y_\delta, \delta) - v(y_\eta, \eta) - (c_j/(n-1)) \sum_{\gamma \in A \setminus A_1} v(y_\delta, \delta) - v(y_\eta, \eta) > 0 \). This inequality holds because \( \delta \in A_1 \) and \( c_j < 1 \).

Recall that \( \alpha \in \text{arg max}_{\gamma \in A} x_\gamma - x'_\gamma \) and \( \mu'_j = \alpha \). Let \( \beta \equiv \mu_j \). We conclude by proving that \( u_j(z) > u_j(z') \), a contradiction. By our claim above and since \( z \) is non-contestable for \( u \), we have that for each \( i \in N \), \( v(z_j) \geq v(z_i) \). Let \( E = A \setminus \{ \alpha, \beta \} \) and \( C \equiv \{ \gamma \in A : v(x'_\gamma, \alpha) \geq v(x'_\gamma, \gamma) \} \). Thus, \( u_j(z) > u_j(z') \) whenever:
\[
\begin{align*}
& \left( 1 - \frac{c_j}{n-1} \right) (v(x_\beta, \beta) - v(x_\alpha, \alpha)) + v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \cap C} (v(x_\beta, \beta) - v(x_\alpha, \alpha) + v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \setminus C} (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& > 0.
\end{align*}
\]
Equivalently,
\[
\begin{align*}
& \left( 1 - \frac{c_j}{n-1} \right) (v(x_\beta, \beta) - v(x_\alpha, \alpha)) + v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \cap C} (v(x_\beta, \beta) - v(x_\alpha, \alpha) + v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \setminus C} (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& > 0.
\end{align*}
\]
Since \( z \) is non-contestable for \( u \), then \( v(x_\beta, \beta) \geq v(x_\alpha, \alpha) \). Since \( |E| = n - 1 \) only if \( \beta = \alpha \) and \( c_j < 1 \), we have that the inequality above holds whenever
\[
\begin{align*}
& v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \cap C} (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& - \frac{c_j}{n-1} \sum_{\gamma \in E \setminus C} (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) \\
& > 0.
\end{align*}
\]
Since for each \( \gamma \notin C \), \( v(x'_\alpha, \alpha) < v(x'_\gamma, \gamma) \), then the inequality above holds whenever
\[
v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) - (c_j/(n-1)) \sum_{\gamma \in E} (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)) > 0.
\]
By the same argument after equation (1) we have that this inequality holds whenever
\[
(v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)) \left( 1 - \frac{c_j}{n-1} \left( |E| + |A \setminus E| + \left( \frac{m-m}{m} \right) 1_{B' \neq \emptyset} |A \setminus B'| \right) \right) > 0,
\]
where \( B' \equiv \{ \gamma \in E : x'_\gamma \geq x_\gamma \} \). This inequality holds by assumption F1. \( \square \)
Proof of Theorem 4. Let \( u \in \mathbb{Z}^n \) be such that for each \( i \in N \) and each pair of different objects \( \alpha \) and \( \beta \), \( c_i(\alpha, \beta) < 1/(n-1) \). Let \( i \in N \). Recall that \( u_i \) has the form: for each \( z \equiv (x, \mu) \in Z \)
\[
  u(z) = v(\mu_i) + x_i + \sum_{j \neq i} c_i(\mu_i, \mu_j)x_j.
\]
To simplify notation we write \( c_i^\beta \) instead of \( c_i(\alpha, \beta) \). Suppose without loss of generality that there are different objects \( \alpha \) and \( \beta \) such that \( c_i^\beta > 0 \), for otherwise the preferences exhibit no externalities, and the result is easily verified. Let \( k \equiv \min_{\alpha \neq \beta, c_i^\beta > 0} 1/c_i^\beta \) and \( \Gamma \equiv \max_{\delta, \gamma \in A} |v(\delta) - v(\gamma)| \). By our hypothesis, \( k > n - 1 \).

We will prove a more general result that will also help us proving Theorem 6. Let \( M \in \mathbb{R} \) and \( b \leq 0 \) be such that \( b < M \). Let \( z \equiv (x, \mu) \in Z_M \) be such that \( x_i = b \) and for each \( j \neq i \), \( x_j \geq b \). Let \( z' \equiv (x', \mu') \) be the allocation obtained by swapping agent \( i \)'s consumption bundle with that of one agent who gets the highest consumption of money at \( z \). We obtain conditions guaranteeing that \( u_i(z') \geq u_i(z) \). Let \( B \equiv \{ \alpha \in A : x_\alpha \leq 0 \} \) and \( C \equiv \{ \alpha \in A : x_\alpha > 0 \} \). Let \( \alpha \equiv \mu_i \) and \( \beta \equiv \mu_i' \). Since \( \alpha \in A \), then \( B \neq \emptyset \). Since \( M > b \), then \( \beta \in C \neq \emptyset \). Denote \( c_i^\alpha = c_i^\beta = 0 \). Then, \( u_i(z') \geq u_i(z) \) whenever:
\[
  v(\alpha) - v(\beta) + b - x_\beta + \sum_{\gamma \in B} (c_i^\alpha - c_i^\beta)x_\gamma + \sum_{\gamma \in C} (c_i^\alpha - c_i^\beta)x_\gamma \leq 0.
\]
This inequality is satisfied whenever \( \Gamma + b - x_\beta - (1/k)\sum_{\gamma \in B} x_\gamma + (1/k)\sum_{\gamma \in C} x_\gamma \leq 0 \).
Since for each \( \gamma \in B \), \( x_\gamma \geq b \) and for each \( \gamma \in C \), \( x_\gamma \leq x_\beta \), the inequality above holds whenever, \( \Gamma + b - x_\beta - |B|b/k + |C|x_\beta/k \leq 0 \). Reorganizing terms, \( \Gamma + (|C| - |B|)b/k \leq (k - |C|)(x_\beta - b)/k \). Now, \( x_\beta - b \geq (M - nb)/(n - 1) \), for otherwise \( M = b + \sum_{\delta \neq \alpha} x_\delta < b+(n-1)b+M-nb = M \). Since \( k > |C| \), then \( u_i(z') \geq u_i(z) \) whenever, \( \Gamma + (|C| - |B|)b/k \leq (k - |C|)(M - nb)/(n - 1) \). Equivalently,
\[
  \Gamma + \left( \frac{(n - 1)(k - |B|) + (k - |C|)}{n - 1} \right)b \leq \left( \frac{k - |C|}{(n - 1)k} \right)M.
\]
Since \( k > |C| \), this inequality is satisfied whenever,
\[
  \frac{(n - 1)k}{k - (n - 1)} \Gamma + \min_{0 \leq |C| \leq n - 1, 0 \leq |B| \leq n - 1, |C| + |C| + |B| = n} \left( \frac{(n - 1)(k - |B|) + (k - |C|)}{k - |C|} \right)b \leq M. \quad (3)
\]
For a fixed \( b \leq 0 \) the left term of the inequality above defines \( M_b^* \) such that for each \( M > \max\{b, M_b^*\} \), \( u_i \) satisfies the compensation assumption at \( b \). Thus, the first part of the theorem follows from Theorem 1.

Finally, we prove that if \( z \in Z_M \) is non-contestable for \( u \), then \( z \) is efficient for \( u \). Let \( M \in \mathbb{R} \) and \( z \equiv (x, \mu) \in Z_M \) be a non-contestable allocation for \( u \). We claim that if \( z' \equiv (x', \mu') \in Z \) is such that \( \sum_{i \in N} x_i' \leq M \) and for each \( i \in N \), \( u_i(z') \geq u_i(z) \), it must
be the case that for each $a \in A$, $x'_a \geq x_a$. This implies that $z'$ is obtained from $z$ by reshuffling consumption bundles, and thus for each $i \in N$, $u_i(z) \geq u_i(z')$. Suppose by contradiction that there is $a \in A$ such that $x'_a < x_a$. Let $a \in \text{arg max}\{x_\beta - x'_\beta : \beta \in A\}$. Let $\epsilon \equiv x_a - x'_a > 0$. Let $i \in N$ be such that $\mu_i = a$. Since $z$ is non-contestable for $u_i$, then $u_i(z) \geq u_i(\hat{z})$, where $\hat{z}$ is obtained from $z$ by swapping the consumption bundle of agent $i$ and the agent who receives $a$. We claim that $u_i(\hat{z}) > u_i(z')$. Writing $c'_a$ for $c_i(\alpha, \beta)$, this is:

$$x_a + \sum_{\beta \in A \setminus \{a\}} c'_\beta x_\beta > x'_a + \sum_{\beta \in A \setminus \{a\}} c'_\beta x'_\beta.$$

Let $B \equiv \{\beta \in A \setminus \{a\} : x_\beta < x'_\beta\}$ and $C \equiv \{\beta \in A \setminus \{a\} : x_\beta \geq x'_\beta\}$. Since $\sum_{\beta \in C} c'_\beta (x_\beta - x'_\beta) \geq 0$, then this inequality is satisfied whenever $(x_a - x'_a) + \sum_{\beta \in B} c'_\beta (x_\beta - x'_\beta) > 0$. Suppose without loss of generality that $B \neq \emptyset$, for otherwise the result is trivial. Let $\Upsilon \equiv \sum_{\beta \in B} c'_\beta (x'_\beta - x_\beta)$. We need to prove that $\Upsilon < x_a - x'_a = \epsilon$. Now, $\Upsilon \leq \max_{\beta \in A \setminus \{a\}} c'_\beta \sum_{\beta \in B} (x'_\beta - x_\beta) < 1/(n - 1) \sum_{\beta \in B} (x'_\beta - x_\beta)$, where the strict inequality follows from $B \neq \emptyset$. Since $\sum_{\beta \in A} x'_\beta \leq \sum_{\beta \in A} x_\beta$, then $\sum_{\beta \in B} (x'_\beta - x_\beta) \leq \sum_{\beta \in A \setminus B} (x_\beta - x'_\beta) \leq |A \setminus B| \epsilon$. Thus, $\Upsilon < (n - 1) \epsilon / (n - 1)$. Thus, $u_i(\hat{z}) > u_i(z') \geq u_i(z) \geq u_i(\hat{z})$. This is a contradiction. \hfill \Box

**Proof of Theorem 6.** Let $M \in \mathbb{R}$ and $u_i \in \mathcal{F}$ such that there is $K > 0$ such that for each pair of objects $a$ and $\beta$ and each $x \in \mathbb{R}$, $|v^i_a(x, a) - v^i_\beta(x, a)| \leq K$. For a fixed $M$, let $b^*$ be such that $M > nb^*$ and for each pair $\{\alpha, \beta\} \subseteq A$, $v^i_\beta(M/n, \beta) - v^i_\alpha(b^*, a) > K/(1 - c_i)$. By the same argument as in Theorem 2, $u_i$ satisfies the compensation assumption at each $b \leq b^*$.

Let $u \in \mathcal{L}^n$ and suppose that for each $i \in N$ and each pair of different objects $a$ and $\beta$, $c_i(a, \beta) < 1/(n - 1)$. Observe that the coefficient that multiplies $b$ in the left term of inequality (3) is positive. Thus, if we fix $M$, there is $b^* < \min\{0, M\}$ such that $u_i$ satisfies the compensation assumption at each $b \leq b^*$.

The theorem then follows from Theorem 5. \hfill \Box

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