Escaping the repugnant conclusion: rank-discounted utilitarianism with variable population*

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Abstract

We contribute to population ethics by proposing and axiomatizing rank-discounted critical-level generalized utilitarianism (RDCLU). Population ethics is needed for evaluation of policies, e.g., concerning climate change, where population size depends on the chosen policy. We show that critical-level generalized utilitarianism and (a version of) critical-level leximin are the limits of RDCLU for extreme values of the rank utility discount factor. Moreover, we establish how RDCLU avoids serious objections raised against other variable population criteria. In particular, it escapes both the Repugnant Conclusion and the Very Sadistic Conclusion (one of which critical-level generalized utilitarianism fails for any critical level).

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1 Introduction

Emission of greenhouse gases may prevent the existence of a great many people who would otherwise have existed, as the earth will not be able to sustain anything approaching our present population if there is extreme climate change (Broome, 2010). How should we take into account the loss of such potential lives when evaluating policies designed to abate greenhouse gas emissions?

Practical evaluation of climate polices in integrated assessment models, like versions of Nordhaus' DICE model (Nordhaus, 2008), usually applies the time-discounted utilitarian criterion, implying that insignificant weight is assigned to people living centuries from now. Since serious climate change will, in particular, reduce population size in the far future, accounting for loss of potential lives is not of great concern under time-discounted utilitarianism.

However, if one instead denounces such deviation from the ideal of equal treatment of all individuals, then the issue of losing potential lives, even many generations from now, becomes much more pressing. If we endorse Broome's (2004) position that one cannot simply ignore the effects on population size when evaluating climate policies, while maintaining that all individuals be treated equally, what kind of evaluation criterion should we use?

One prominent response is to assume that there exists a critical level of lifetime wellbeing which, if experienced by an added individual without changing the wellbeing levels of the existing population, leads to an alternative which is as good as the original. This critical level may differ from the (individually) *neutral* level of wellbeing, above which a life is worth living, and below which it is not. Combined with a generalized utilitarian criterion where, for fixed population size,

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the undiscounted of the sum of the individuals' (transformed) utilities is maximized, the idea of a critical level leads to critical-level generalized utilitarianism (Broome, 2004; Blackorby, Bossert and Donaldson, 2005).

The standard objection to critical-level generalized utilitarianism is that

- if we set the critical level at (or below) the individually neutral wellbeing level, then we get the Repugnant Conclusion (Parfit, 1976, 1982, 1984) where, for any population with excellent lives, there is a population with lives barely worth living that is better, provided that the latter includes sufficiently many people,

- if we set the critical level strictly above the individually neutral wellbeing level, then we get the Very Sadistic Conclusion (Arrhenius, 2000, 2014) where, for any population with terrible lives not worth living, there is a population with good lives that is worse, provided that the latter includes sufficiently many people.

There are other criteria that avoid the Repugnant and Very Sadistic Conclusions, but they all have their own serious shortcomings. According to average generalized utilitarianism, the contributive value of a life can vary in all respects: a life worth living can have negative contributive value and a life not worth living can have positive contributive value. According to critical-level leximin, as defined by Blackorby, Bossert and Donaldson (1996), any population with excellent lives is worse than a population with one added individual even when the wellbeing of all the individuals in the latter population is barely above the critical level. According to critical-level leximin, as suggested by Arrhenius (2014, Sect. 6.8) and defined in Section 2 of the present paper (Definition 5), any population is worse than a population consisting of one individual, provided that the worst-off individual of the former has lower wellbeing than the single individual of the latter.

Even in a fixed population framework, where there is no difference between the various generalized utilitarian criteria discussed above, and also no difference between the two kinds of leximin criteria, both generalized utilitarianism and leximin have shortcomings if there are many present and future people. If one considers a completely egalitarian wellbeing stream in an intergenerational setting with many future generations, and seeks to evaluate whether the present generation should make a sacrifice leading to a uniform benefit of all future generations, then generalized utilitarianism and leximin reach opposite and extreme conclusions: According to generalized utilitarianism, the sacrifice should always be made provided that there are sufficiently many future generations, while according to leximin, it should never be made (cf. Asheim, 2010, Section 4.3).

As a response, Zuber and Asheim (2012) have proposed and axiomatized the rank-discounted generalized utilitarian social welfare order where the weights assigned to (transformed) utility are discounted according to rank, not according to time as in the time-discounted utilitarian criterion. Such rank-discounting is solely an expression of inequality aversion and is therefore consistent with equal treatment. Still, it allows for a trade-off between the present and the future in the conflict described in the previous paragraph, and leads to the same consequences as time-discounted utilitarianism if wellbeing is perfectly correlated with time. However, if a future generation is expected to be worse-off compared to the present, e.g. due to climate change, then that less fortunate future generation will not be assigned less utility weight than the present according to rank-discounted utilitarianism. It will actually receive strictly more utility weight provided there are not infinitely many even worse-off generations. Here we show how the rank-discounted generalized utilitarian approach can fruitfully be extended to the variable population framework. In particular, it avoids shortcomings like the Repugnant and Very Sadistic Conclusions.

In Section 2 we propose the *rank-discounted critical level generalized utilitarian* (RDCLU) social welfare order (Definition 1). As is the case for Ng's (1989) theory X and Sider's (1991) principle GV, an RDCLU social welfare order is a *variable value principle*, in the sense that the value of an egalitarian population does not change affinely with population size, and a *context sensitive theory*, in the sense that the contributive value of a life depends on the wellbeing of the rest of the population. However, in contrast to Ng's theory X, the value of each individual depends on its rank, and in contrast to Sider's principle GV, the values are assigned in a prioritarian manner. Then, in Section 3, we provide an axiomatic foundation and show how critical-level generalized utilitarianism (Definition 2) and critical-level leximin (as defined in Definition 5) are the limits of an RDCLU social welfare order for extreme values of the rank utility discount factor.

In Section 4 we present many of the conclusions and principles used to evaluate variable population criteria (cf. Arrhenius, 2014), and establish how the RDCLU social welfare order avoids the serious objections raised against other variable population criteria. In particular, it escapes both the Repugnant Conclusion and the Very Sadistic Conclusion (one of which critical-level generalized utilitarianism fails for any critical level), it never assigns positive contributive value to a life not worth living (in contrast to average generalized utilitarianism) and it avoids the Weak Repugnant and Reverse Repugnant Conclusions (which are the conclusions that, in more extreme versions, the two kinds of critical-level leximin lead to). The key intuition is that, with an RDCLU social welfare order, the cumulative effect of adding individuals at a given level of lifetime wellbeing is bounded.

Finally, in Section 5 we illustrate in the context of a simple model of optimal population size how the RDCLU social welfare order leads to a smaller population size than critical-level generalized utilitarianism and, in Section 6, note that the problem of accommodating uncertainty still remains. Proofs are contained in an

appendix.

2 Framework and definitions

Let $\mathbb{N}$ denote the natural numbers, let $\mathbb{R}$ denote the real numbers, and let respectively $\mathbb{R}_+/\mathbb{R}_{++}/\mathbb{R}_-/\mathbb{R}_{--}$ denote the non-negative/positive/non-positive/negative real numbers. Let $\mathbf{X} = \cup_{n\in\mathbb{N}}\mathbb{R}^n$ be the set of possible finite *allocations* of lifetime wellbeing. For every $n \in \mathbb{N}$, each allocation $\mathbf{x} \in \mathbb{R}^n$ determines the finite population size, $n(\mathbf{x}) = n$, and the distribution of wellbeing, $\mathbf{x} = (x_1, \ldots, x_{n(\mathbf{x})})$, among the $n(\mathbf{x})$ individuals that make up the population. This framework suffices, as the binary relations we consider do not depend on the identities of the individuals. Following the usual convention in population ethics, lifetime wellbeing equal to 0 represents *neutrality*. Hence, lifetime wellbeing is normalized so that above neutrality, a life, as a whole, is worth living; below neutrality, it is not.

A social welfare relation (SWR) on the set $\mathbf{X}$ is a binary relation $\succsim$, where for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ implies that the allocation $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$. A complete, reflexive and transitive SWR is called a social welfare order (SWO).

For each $\mathbf{x} \in \mathbf{X}$, let $\mathbf{x}_{[]} = (x_{[1]}, \ldots, x_{[r]}, \ldots, x_{[n(\mathbf{x})]})$ denote the non-decreasing allocation which is a reordering of $\mathbf{x}$; i.e., for each rank $r \in \{1, \ldots, n(\mathbf{x}) - 1\}$, $x_{[r]} \leq x_{[r+1]}$. Even though the permutation that turns $\mathbf{x}$ into $\mathbf{x}_{[]}$ need not be uniquely determined if different individuals have the same lifetime wellbeing, the resulting rank-ordered allocation, $\mathbf{x}_{[]}$, is uniquely determined.

For every $n \in \mathbb{N}$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, write $\mathbf{x}_{[]} \geq \mathbf{y}_{[]}$ whenever $x_{[r]} \geq y_{[r]}$ for all $r \in \{1, \ldots, n(\mathbf{x})\}$, and write $\mathbf{x}_{[]} > \mathbf{y}_{[]}$ whenever $\mathbf{x}_{[]} \geq \mathbf{y}_{[]}$ and $\mathbf{x}_{[]} \neq \mathbf{y}_{[]}$. Let $(z)_n \in \mathbb{R}^n$ denote the *egalitarian* allocation where all n individual have lifetime wellbeing equal to z. Let $(\mathbf{x}, z)$ denote $\mathbf{x} \in \mathbf{X}$ with one added individual with lifetime wellbeing equal to $z \in \mathbb{R}$. Let $\big(\mathbf{x}, (z)_n\big)$ denote $\mathbf{x} \in \mathbf{X}$ with n added

individuals, all with lifetime wellbeing equal to $z \in \mathbb{R}$.

We now introduce the SWO proposed, axiomatized and analyzed in this paper.

Definition 1 An SWR $\succsim$ on $\mathbf{X}$ is a *rank-discounted critical-level generalized utilitarian* (RDCLU) SWO if there exist $c \in \mathbb{R}_+$, $\beta \in (0,1)$, and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \sum\nolimits_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \;\geq\; \sum\nolimits_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) \;.$$

This is a *generalized* utilitarian criterion, as the function u turns lifetime wellbeing into transformed values. We refer to the transformed values as *utility* and β as a *rank utility discount factor* to be consistent with the literature on intertemporal social choice. The term 'rank-discounted' reflects that the utility weights are not merely 'rank-dependent', but are discounted according to rank by a geometrically decaying function. This is analogous to the use of 'time-discounted utilities', rather than 'time-dependent utilities', in the time-discounted utilitarian criterion of intertemporal social choice. We refer to c as a *critical-level parameter*.

The following are established generalized utilitarian SWOs of population ethics.

Definition 2 An SWR $\succsim$ on $\mathbf{X}$ is a *critical-level generalized utilitarian* (CLU) SWO if there exist $c \in \mathbb{R}_+$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \sum\nolimits_{r=1}^{n(\mathbf{x})} \left(u(x_{[r]}) - u(c)\right) \;\geq\; \sum\nolimits_{r=1}^{n(\mathbf{y})} \left(u(y_{[r]}) - u(c)\right) \;.$$

The CLU SWO has been proposed, discussed and axiomatized in Blackorby and Donaldson (1984) and Blackorby, Bossert and Donaldson (1995, 2005). The CLU SWO with $c = 0$ is referred to as the *total generalized utilitarian* (TU) SWO.

Definition 3 An SWR $\succsim$ on $\mathbf{X}$ is an *average generalized utilitarian* (AU) SWO if there exists a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \tfrac{1}{n(\mathbf{x})}\sum\nolimits_{r=1}^{n(\mathbf{x})} u(x_{[r]}) \;\geq\; \tfrac{1}{n(\mathbf{y})}\sum\nolimits_{r=1}^{n(\mathbf{y})} u(y_{[r]})\,.$$

The AU SWO is not affected by how $u(0)$ is normalized, and thus not affected by the introduction of a critical-level parameter c either.

Definition 4 An SWR $\succsim$ on $\mathbf{X}$ is a *number-dampened critical-level generalized utilitarian* (NDCLU) SWO if there exist $c \in \mathbb{R}_+$, a function $f : \mathbb{N} \to \mathbb{R}_{++}$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \tfrac{f(n(\mathbf{x}))}{n(\mathbf{x})}\sum\nolimits_{r=1}^{n(\mathbf{x})} \big(u(x_{[r]}) - u(c)\big) \;\geq\; \tfrac{f(n(\mathbf{y}))}{n(\mathbf{y})}\sum\nolimits_{r=1}^{n(\mathbf{y})} \big(u(y_{[r]}) - u(c)\big)\;.$$

The NDCLU SWO has been introduced in the case with $c = 0$ by Ng (1989). It encompasses both the CLU SWO when $f(n) = n$ for all $n \in \mathbb{N}$, and the AU SWO when $f(n) = 1$ for all $n \in \mathbb{N}$.

Lastly, we state a version of a leximin SWO.

Definition 5 An SWR $\succsim$ on $\mathbf{X}$ is a *critical-level leximin* (CLL) SWO if there exists $c \in \mathbb{R}_+$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) \geq n(\mathbf{y})$,

(a) $\mathbf{x} \sim \mathbf{y}$ if and only if $(x_{[1]}, \ldots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{y})+1]}, \ldots, x_{[n(\mathbf{x})]}) = (c)_{n(\mathbf{x})-n(\mathbf{y})}$, and

(b) $\mathbf{x} \succ \mathbf{y}$ if and only if (i) there exists $R \in \{1, \ldots, n(\mathbf{y})\}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > y_{[R]}$ or (ii) $(x_{[1]}, \ldots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{y})+1]}, \ldots, x_{[n(\mathbf{x})]}) > (c)_{n(\mathbf{x})-n(\mathbf{y})}$.

The CLL SWO is suggested by Arrhenius (2014, Sect. 6.8). It differs, in comparisons of allocations with different population sizes, from the critical-level leximin SWO proposed and axiomatized by Blackorby, Bossert and Donaldson (1996).[1]

[1]Their critical-level leximin SWO is defined as follows: For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) \geq n(\mathbf{y})$

3 Axioms and representation results

Rank-discounted critical-level generalized utilitarianism can be characterized by the following seven axioms. The first three axioms are sufficient to ensure numerical representation for any fixed population size, and entail that individuals are treated anonymously and with sensitivity for their well-being.

Axiom 1 (Order) *The relation* $\succsim$ *is complete, reflexive and transitive on* $\mathbf{X}$.

Axiom 2 (Continuity) *For all* $n \in \mathbb{N}$ *and* $\mathbf{x} \in \mathbb{R}^n$*, the sets* $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} \succsim \mathbf{x}\}$ *and* $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} \succsim \mathbf{y}\}$ *are closed.*

Axiom 3 (Suppes-Sen) *For all* $n \in \mathbb{N}$ *and* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$*, if* $\mathbf{x}_{[]} > \mathbf{y}_{[]}$*, then* $\mathbf{x} \succ \mathbf{y}$.

While ordinary critical-level generalized utilitarianism allows for unrestricted independence to adding an individual (see Blackorby, Bossert and Donaldson, 2005), our axioms impose such independence only if the added individual is best-off (relative to two allocations with the same population size) or worst-off.

Axiom 4 (Existence independence of the best-off) *For all* $n \in \mathbb{N}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ *and* $z \in \mathbb{R}$ *satisfying* $z \geq \max\{x_{[n]}, y_{[n]}\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z)$ *if and only if* $\mathbf{x} \succsim \mathbf{y}$.

Axiom 5 (Existence independence of the worst-off) *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and* $z \in \mathbb{R}$ *satisfying* $z \leq \min\{x_{[1]}, y_{[1]}\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z)$ *if and only if* $\mathbf{x} \succsim \mathbf{y}$.

In the spirit of critical-level generalized utilitarianism, we introduce a critical lifetime wellbeing $c \in \mathbb{R}_+$, which if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original. However, the following axiom imposes this if $x_{n(\mathbf{x})} \leq c$, not otherwise. Note that c is at least as large as neutral lifetime wellbeing.

(where $\tilde{\mathbf{y}}$ denotes $\big(\mathbf{y}, (c)_{n(\mathbf{x})-n(\mathbf{y})}\big)$), (a) $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x}_{[]} = \tilde{\mathbf{y}}_{[]}$, and (b) $\mathbf{x} \succ \mathbf{y}$ if and only if there exists $R \in \{1, \ldots, n(\mathbf{x})\}$ such that $x_{[r]} = \tilde{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > \tilde{y}_{[R]}$.

Axiom 6 (Existence of a critical level) *There exist $c \in \mathbb{R}_+$ and $n \in \mathbb{N}$ such that, for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $x_{[n]} \leq c$, $(\mathbf{x}, c) \sim \mathbf{x}$.*

All axioms above are satisfied also by ordinary critical-level generalized utilitarianism. However, as mentioned in the introduction and discussed by Arrhenius (2014, Sect. 5.1), the CLU SWO leads to the Repugnant Conclusion if $c = 0$ (adding sufficiently many individuals with lifetime wellbeing just above 0 makes the allocation better than any fixed alternative), and leads to the Very Sadistic Conclusion if $c > 0$ (adding sufficiently many individuals with positive lifetime wellbeing just below c makes the allocation worse than any fixed alternative). Both conclusions rely on the property that adding individuals at a given utility level has unbounded importance. One may consider this property too extreme, and rather impose through the following axiom that this need not be the case.

Axiom 7 (Existence of egalitarian equivalence) *For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \succ \mathbf{y}$, then there exists $z \in \mathbb{R}$ such that, for all $N \in \mathbb{N}$, $\mathbf{x} \succ (z)_n \succ \mathbf{y}$ for some $n \geq N$.*

Axiom 7 is the key axiom to avoiding the Repugnant and Very Sadistic Conclusions, while not by itself contradicting these conclusions, thus weaker than directly requiring to avoid the conclusions. For instance, the NDCLU SWO with $f(n) = 1$ if n is odd and $f(n) = n$ if n is even satisfies Axiom 7 while leading to the Repugnant Conclusion if $c = 0$ and the Very Sadistic Conclusion if $c > 0$.

We now state the result (proven in the appendix) that these seven axioms characterize the rank-discounted critical-level generalized utilitarian SWO.

Theorem 1 *Consider an SWR $\succsim$ on $\mathbf{X}$. The following two statements are equivalent.*

(1) $\succsim$ satisfies Axioms 1–7.

(2) $\succsim$ is an RDCLU SWO.

It follows from the RDCLU SWO that c is the lifetime wellbeing which, if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original *only if* $x_{[n(\mathbf{x})]} \leq c$. If $x_{[n(\mathbf{x})]} > c$, then there is a context-dependent critical level in the open interval $(c, x_{[n(\mathbf{x})]})$ which depends on the wellbeing levels that exceed c. This follows from Definition 1, since adding an individual with wellbeing equal to $x_{[n(\mathbf{x})]}$ increases welfare, while adding an individual with wellbeing equal to c lowers the weights assigned to individuals at wellbeing levels that exceed c and thereby reduces welfare.

We end this section by showing how any CLU and CLL SWO is the limit of an RDCLU SWO for extreme values of the rank utility discount factor β. Write $\succsim_{\beta,u,c}$ for the RDCLU SWO characterized by β, u and c, $\succsim_{u,c}$ for the CLU SWO characterized by u and c, and $\succsim_c^L$ for the CLL SWO characterized by c. The following result (proven in the appendix) establishes that, for any non-negative c and any increasing and continuous function u, the CLU SWO $\succsim_{u,c}$ is the limit of the RDCLU SWO $\succsim_{\beta,u,c}$ as β approaches 1 and the CLL SWO $\succsim_c^L$ is the limit of the RDCLU SWO $\succsim_{\beta,u,c}$ as β approaches 0. Note that in the case of $\succsim_c^L$, the weak preference of $\succsim_{\beta,u,c}$ for small β is both sufficient and necessary.

Theorem 2 *For any $c \in \mathbb{R}_+$ and any continuous and increasing function u, the following results hold: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,*

(i) $\mathbf{x} \succsim_{u,c} \mathbf{y}$ *if there exists $\bar{\beta} \in (0,1)$ such that, for all $\beta \in (\bar{\beta}, 1)$, $\mathbf{x} \succsim_{\beta,u,c} \mathbf{y}$.*

(ii) $\mathbf{x} \succsim_c^L \mathbf{y}$ *if and only if there exists $\underline{\beta} \in (0,1)$ such that, for all $\beta \in (0, \underline{\beta})$, $\mathbf{x} \succsim_{\beta,u,c} \mathbf{y}$.*

The case where the rank-discount factor β approaches 0, and thereby an RDCLU SWO approaches a CLL SWO, is related to the case where a given allocation is replicated in the following sense: For any $\mathbf{x} \in \mathbf{X}$ and any $k \in \mathbb{N}$,

the *k-replica* of $\mathbf{x}$ is an allocation $\mathbf{x}^k$ with $kn(\mathbf{x})$ individuals having the property that $x_{[r]} = x^k_{[\rho]}$ for all $r \in \{1, \ldots, n(\mathbf{x})\}$ and $\rho \in \{k(r-1)+1, \ldots, kr\}$. Theorem 2 implies that, for fixed $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $\beta \in (0,1)$, there exists $K \in \mathbb{N}$ such that, for all $k > K$, $\mathbf{x}^k \succsim_{\beta,u,c} \mathbf{y}^k$ if and only if $\mathbf{x} \succsim^L_c \mathbf{y}$. Hence, as a given allocation is replicated, utility weight is redistributed towards the individuals with lowest lifetime wellbeing.

4 Evaluating RDCLU

We now compare the RDCLU SWO to other criteria by means of conditions used in the literature on population ethics. In particular, we establish two results that show how the RDCLU SWO compares favorably to the NDCLU SWO for any function f (cf. Definition 4). The conditions are referred to by the labels used by Arrhenius (2014), but are formulated here in a precise manner using the terminology of this paper.

According to Parfit (1976, 1982, 1984), an SWR leads to the Repugnant Conclusion if for any egalitarian allocation with very high positive wellbeing, there is an egalitarian allocation with very low positive wellbeing which is better.

> An SWR $\succsim$ leads to the *Repugnant Conclusion* if, for all y, $z \in \mathbb{R}$ with $y > z > 0$ and $k \in \mathbb{N}$, there is $n > k$ such that $(y)_k \prec (z)_n$.[2]

The TU SWO has been criticized for not avoiding this conclusion.

[2]This definition is equivalent to the one provided by Blackorby, Bossert and Donaldson (2005, p. 162). If one determines disjoint non-empty intervals $(0, x^\ell]$ and $[x^h, \infty)$ with, respectively, very low and very high positive wellbeing, then the Repugnant Conclusion might alternatively be defined as the property that, for all $y \in [x^h, \infty)$ and $k \in \mathbb{N}$, there are $z \in (0, x^\ell]$ and $n > k$ such that $(y)_k \prec (z)_n$. With this alternative definition, the CLU SWO avoids the Repugnant Conclusion if $c \geq x^\ell$, but not if $c \in (0, x^\ell)$, while the RDCLU SWO still avoids the Repugnant Conclusion for all $c \geq 0$ (just pick k large enough to ensure $(1-\beta^k)u(x^h) \geq u(x^\ell)$).

Besides, if $c > 0$, the CLU SWO is subject to the following weaker form of the Repugnant Conclusion, namely that for any egalitarian allocation with very high positive wellbeing, there is an egalitarian allocation with wellbeing just above the critical level which is better:

> An SWR $\succsim$ leads to the *Weak Repugnant Conclusion* if, for all y, $z \in \mathbb{R}$ with $y > z > c$ and $k \in \mathbb{N}$, there is $n > k$ such that $(y)_k \prec (z)_n$.

As pointed out by (Arrhenius, 2000, 2014), the CLU SWO with $c > 0$ also leads to the Very Sadistic Conclusion that, for any allocation with negative wellbeing, there is an egalitarian allocation with positive wellbeing which is worse.

> An SWR $\succsim$ leads to the *Very Sadistic Conclusion* if, for all $\mathbf{x} \in \mathbf{X}$, there are $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$ such that $\mathbf{x} \succ (z)_n$.

This follows since, if $0 < z < c$, then by choosing n sufficiently large, $(z)_n$ can be made worse than any given allocation $\mathbf{x}$, even when the individuals in $\mathbf{x}$ have very negative lifetime wellbeing.

As an alternative to avoiding the Very Sadistic Conclusion, Arrhenius (2014) has also proposed the Weak Non-Sadism Condition:

> An SWR $\succsim$ satisfies the *Weak Non-Sadism Condition* if there are $y \in \mathbb{R}_{--}$ and $k \in \mathbb{N}$ such that, for all $\mathbf{x} \in \mathbf{X}$, $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$, $\big(\mathbf{x}, (y)_k\big) \precsim \big(\mathbf{x}, (z)_n\big)$.

The RDCLU avoids the Weak Repugnant Conclusion, even when $c = 0$, implying that it also escapes the Repugnant Conclusion for all $c \geq 0$. It also avoids the Very Sadistic Conclusion and, provided that the function u is bounded above, satisfies the Weak Non-Sadism Condition. This is in contrast with the large class of generalized utilitarian criteria embodied in the NDCLU SWO: these criteria necessarily violate the Weak Non-Sadism Condition or imply the Weak Repugnant Conclusion, as stated in the following result (proven in the appendix).

Proposition 1

(i) An NDCLU SWO either leads to the Weak Repugnant Conclusion or violates the Weak Non-Sadism Condition, even when the function u is bounded above.

(ii) An RDCLU SWO avoids the Weak Repugnant Conclusion and satisfies the Weak Non-Sadism Condition, provided the function u is bounded above.

Being able to avoid the Weak Repugnant Conclusion while satisfying the Weak Non-Sadism Condition, the RDCLU SWO is rather remarkable. The NDCLU SWO is indeed one of the most general forms that have been proposed.[3]

Outside the class of generalized utilitarian criteria, Blackorby, Bossert and Donaldson (1996) have proposed a critical-level leximin SWO (cf. footnote 1). It however leads to the following more extreme version of the Weak Repugnant Conclusion: For all $y, z \in \mathbb{R}$ with $y > z > c$ and $k, n \in \mathbb{N}$ with $k < n$, $(y)_k \prec (z)_n$. Thus it does not escape the impossibility encountered by the NDCLU SWO.

A version of Proposition 1 that restricts attention to the more undesirable implications of variable population criteria – the Repugnant and Very Sadistic Conclusions – is obtained if one adopts the following natural requirement (called *Negative Expansion Principle* by Blackorby, Bossert and Donaldson, 2005):

[3] A restricted number-dampened generalized utilitarian (RNDU) SWO is attributed to Thomas Hurka by Blackorby, Bossert and Donaldson (2005). It is similar to the NDCLU SWO with $c = 0$, except that $f(n) = n$ if average wellbeing is negative. Considering the RNDU SWO would not alter our result, because the proof of Proposition 1 only involves streams with positive average wellbeing. Another family is number-sensitive critical-level generalized utilitarianism (Blackorby, Bossert and Donaldson, 2002, 2005). It is similar to the CLU except that the critical level changes with population size. It leads to the Very Sadistic Conclusion and a modified version of the Weak Repugnant Conclusion.

An SWR $\succsim$ satisfies the *Negative Mere Addition Principle* if, for all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}_{--}$, $(\mathbf{x}, z) \prec \mathbf{x}$.

This property is satisfied by the RDCLU SWO, but not by some utilitarian criteria like the AU SWO. And we have the following result (proven in the appendix).

Proposition 2

(i) An NDCLU SWO satisfying the Negative Mere Addition Principle leads either to the Repugnant Conclusion or to the Very Sadistic Conclusion.

(ii) An RDCLU SWO avoids the Repugnant Conclusion and the Very Sadistic Conclusion, and satisfies the Negative Mere Addition Principle.

Contrary to Proposition 1, Proposition 2 does not depend on the function u being bounded above for the RDCLU SWO to satisfy all desirable properties.

Besides the RDCLU SWO, the CLL SWO also avoids all the above shortcomings. However, the CLL SWO leads to the problematic conclusion that for any egalitarian allocation with very high positive wellbeing, there is a better one-individual allocation with slightly higher wellbeing:

An SWR $\succsim$ leads to the *Reverse Repugnant Conclusion* if, for all $y \in \mathbb{R}_{++}$ and $k \in \mathbb{N}$, $(y)_k \prec (z)_1$ if $y < z$.

This conclusion does not follow from the RDCLU SWO as

$$\tfrac{1-\beta^k}{1-\beta}\left(u(y) - u(c)\right) \geq \left(u(z) - u(c)\right),$$

and thus $(y)_k \succsim (z)_1$ according to the RDCLU SWO, is clearly consistent with $c < y < z$, provided that z is sufficiently close to y and k is sufficiently large. The conclusion does not follow from the CLU SWO either, but it follows from the AU SWO. The CLL SWO leads even to the following more extreme version

of the Reverse Repugnant Conclusion: For all $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \prec (z)_1$ if $x_{[1]} < z$. We believe that this disqualifies the CLL SWO as a satisfying criterion.

Propositions 1 and 2 build a strong case in favor of the of the RDCLU SWO. It is however interesting to further assess the criterion with respect to other principles proposed in the literature on variable population criteria.

As shown in Arrhenius (2014, Sect. 3.8), both the TU and AU SWOs fail the following principle:

> An SWR $\succsim$ satisfies the *Strong Quality Addition Principle* if there are $y,\ z \in \mathbb{R}$ with $y > z > 0$ and $k \in \mathbb{N}$ such that, for all $\mathbf{x} \in \mathbf{X}$ and $n > k$, $\big(\mathbf{x}, (y)_k\big) \succsim \big(\mathbf{x}, (z)_n\big)$.

In contrast, this principle is satisfied by the CLL SWO and also by the CLU SWO with $c > 0$ (just let $y > c > z > 0$, implying that $\big(\mathbf{x}, (y)_k\big) \succ \big(\mathbf{x}, (z)_n\big)$ for all $\mathbf{x} \in \mathbf{X}$ and $k,\ n \in \mathbb{N}$ according to the CLU SWO).

To show that the Strong Quality Addition Principle is satisfied by the RDCLU SWO with $c \geq 0$ (i.e., even if $c = 0$) is more delicate: Let $y,\ z \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfy the requirements that $y > z > c$ and

$$\tfrac{1-\beta^k}{1-\beta}\left(u(y) - u(c)\right) \geq \tfrac{1}{1-\beta}\left(u(z) - u(c)\right).$$

Then clearly $\big(\mathbf{x}, (y)_k\big) \succ \big(\mathbf{x}, (z)_n\big)$ according to the RDCLU SWO for all $n \in \mathbb{N}$ if $x_{[n(\mathbf{x})]} \leq z$. Moreover, with $n > k$, raising elements of $\mathbf{x}$ above z contributes more to $\big(\mathbf{x}, (y)_k\big)$ than to $\big(\mathbf{x}, (z)_n\big)$ as their ranks are lower in the former allocation than in the latter, implying that the strict preference cannot be reversed.

Arrhenius (2014) suggests the following weak version of the Pigou-Dalton transfer principle, which is satisfied if any given allocation is deemed worse than an egalitarian allocation with the same number of individuals and higher average (and thus higher total) wellbeing.

An SWR $\succsim$ satisfies the *Non-Anti Egalitarianism Principle* if, for all $\mathbf{x} \in \mathbf{X}$ and $z > (1/n(\mathbf{x})) \sum_{r=1}^{n(\mathbf{x})} x_{[r]}$, $\mathbf{x} \prec (z)_{n(\mathbf{x})}$.

This principle is satisfied by the RDCLU SWO if u satisfies $\beta \times \mathcal{C}_u \leq 1$, where

$$\mathcal{C}_u = \sup_{0<\varepsilon\leq x\leq x'} \frac{u\big(x'+\varepsilon\big) - u\big(x'\big)}{u\big(x\big) - u\big(x-\varepsilon\big)}$$

is an index of non-concavity of the function u.[4] It is also satisfied by the CLU and AU SWOs if u is concave and, of course, by the CLL SWO.

Lastly, the RDCLU, CLU (if $c > 0$), AU and CLL SWOs all violate the following principle:

An SWR $\succsim$ satisfies the *Mere Addition Principle* if, for all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}_{++}$, $(\mathbf{x}, z) \succsim \mathbf{x}$.

Of the SWRs we consider, only the NDCLU SWO with $c = 0$ and for a restricted set of the function f, including the TU SWO, satisfies the Mere Addition Principle. The RDCLU SWO with $c = 0$ does not, because adding an individual with low positive wellbeing will decrease the weights on individuals with higher wellbeing and might thereby worsen the allocation.[5]

With the possible exception of the Mere Addition Principle, this shows that the RDCLU SWO has desirable properties when evaluated by the above conditions and principles. These positive conclusions are of interest in view of Arrhenius (2014, Sect. 11.14), where it is shown that the Weak Non-Sadism Condition is hard to satisfy unless other desirable properties are given up.

[4]See the proof of Proposition 6 in Zuber and Asheim (2012).

[5]Indeed, adding people with positive but very low well-being may increase inequality, an objection which is often made against the Mere Addition Principle. See for instance Carlson (1998), who also proves that the Mere Addition Principle and the Non-Anti Egalitarianism Principle imply a conclusion similar to the Repugnant Conclusion.

Table 1: Population principles and social welfare orders

	RDCLU $c \geq 0$	CLU $c = 0$	CLU $c > 0$	AU	CLL $c \geq 0$
Avoiding the Repugnant Conclusion	$+$	$-$	$+$	$+$	$+$
Avoiding the Weak Repugnant Conclusion	$+$	$-$	$-$	$+$	$+$
Avoiding the Very Sadistic Conclusion	$+$	$+$	$-$	$+$	$+$
Weak Non-Sadism Condition	$+^{a}$	$+$	$-$	$-$	$+$
Negative Mere Addition Principle	$+$	$+$	$+$	$-$	$+$
Avoiding the Reverse Repugnant Conclusion	$+$	$+$	$+$	$-$	$-$
Strong Quality Addition Principle	$+$	$-$	$+$	$-$	$+$
Non-Anti Egalitarianism Principle	$+^{b}$	$+^{c}$	$+^{c}$	$+^{c}$	$+$
Mere Addition Principle	$-$	$+$	$-$	$-$	$-$

(a) If $\{u(z) : z \in \mathbb{R}\}$ is bounded above. (b) If $\beta \times \mathcal{C}_u \leq 1$. (c) If u is concave.

Table 1 summarizes the different principles and conclusions satisfied by the different SWOs discussed in this section, where the sign '+' denotes that the principle is satisfied (or the conclusion follows or, in the first three lines and the sixth, the conclusion is avoided) and the sign '−' denotes the opposite.

5 Optimal population size

Following Dasgupta (1988, pp. 123–125), let m be the total available amount of a consumption good, and let, as before, n denote the number of individuals. Let the wellbeing of each individual be equal to allocated consumption minus s, implying that s is the level of consumption needed to attain neutrality. Hence, a life is worth living if consumption exceeds s, while it is not if consumption falls below s.

Under the AU SWO, the optimal population size n is equal to 1, as this maximizes average utility. Likewise for the CLL SWO, as this maximizes the utility of the worst-off individual. Turn now to the CLU and RDCLU SWOs. Under the assumption that u is concave, it is optimal to divide the available amount m equally among the n individuals. Hence, the so-called *genesis problem* is to optimize n given that each individual's wellbeing $x(n)$ equals $(m/n) - s$, with n treated as a continuous variable, for tractability. Dasgupta (2005) argues that the genesis problem might not be the most interesting problem for population ethics. It is also different from the problem studied by Palivos and Yip (1993) and Razin and Yuen (1995), where the development of per capita wellbeing and population size is optimized within models of economic growth. Still, it is illustrative and leads to generalizable insights (cf. Dasgupta, 1988, fn. 16).

Under the CLU SWO, the genesis problem becomes

$$\max_n n \left(u \left(\tfrac{m}{n} - s \right) - u(c) \right) ,$$

leading to the first-order condition

$$u(x(n)) - u(c) = (x(n) + s) u'(x(n)) . \tag{1}$$

If $u(x) = (1/(1-\eta)) (x+s)^{1-\eta}$ with $\eta > 1$, then (1) can be transformed to

$$\frac{x(n) + s}{c + s} = \eta^{\frac{1}{\eta - 1}} .$$

As the elasticity of marginal utility η goes to infinity, $\eta^{1/(\eta-1)}$ goes to unity, illustrating how the CLU SWO leads to the Repugnant Conclusion if $c = 0$ and to the Weak Repugnant Conclusion otherwise.

Under the RDCLU SWO, the genesis problem becomes

$$\max_n \tfrac{1-\beta^n}{1-\beta} \left(u \left(\tfrac{m}{n} - s \right) - u(c) \right) ,$$

leading to the first-order condition

$$\gamma(n, \beta) \left(u(x(n)) - u(c) \right) = (x(n) + s) u'(x(n)) , \tag{2}$$

where

$$\gamma(n,\beta) := \frac{\beta^n(-\ln\beta^n)}{1-\beta^n}$$

can be shown to satisfy $0 < \gamma(n,\beta) < 1$, $\partial\gamma/\partial n < 0$, $\partial\gamma/\partial\beta > 0$ and $\lim_{\beta\to 1}\gamma(n,\beta) = 1$. If $u(x) = (1/(1-\eta))\,(x+s)^{1-\eta}$ with $\eta > 1$, then (2) can be transformed to

$$\frac{x(n)+s}{c+s} = \left(\frac{\gamma(n,\beta)+\eta-1}{\gamma(n,\beta)}\right)^{\frac{1}{\eta-1}} .$$

The l.h.s. is a decreasing function of n which equals 1 for $n = m/(c+s)$ and approaches ∞ as $n \downarrow 0$. The r.h.s. is greater than $\eta^{1/(\eta-1)} > 1$ and an increasing function of n — implying that the first-order condition determines a unique optimal value of n — and a decreasing function of β which approaches $\eta^{1/(\eta-1)}$ as $\beta \to 1$ — implying that this optimal population size is lower under the RDCLU SWO than under the CLU SWO. Thus, this analysis illustrates how the RDCLU SWO leads to an escape from the Repugnant Conclusion.

6 Concluding remarks

We have contributed to population ethics by proposing and axiomatizing the rank-discounted critical-level generalized utilitarian (RDCLU) SWO. By doing so we have taken one step towards preparing the rank-discounted utilitarian criterion (see Zuber and Asheim, 2012) for practical use.

First of all, we have generalized rank-discounted utilitarianism by letting individuals rather than generations be the object of analysis. This generalization has several implications, one of which is particularly interesting to point out: If there is no intragenerational inequality and per capita wellbeing increases over time, then the aggregate marginal utility of a generation increases with the number of individuals belonging to this generation. On the other hand, the average rank-dependent discount rate with which this aggregate marginal utility is discounted between this generation and its immediate predecessor increases with its size.

Secondly, we have allowed for analysis of a situation where population changes endogenously, e.g., as a consequence of climate change. By introducing a critical level which if experienced by an added individual without changing the utilities of the existing population leads to an alternative which is as good as the original *only if* the wellbeing levels of the existing population does not exceed the critical level, we have been able to combine critical-level population ethics with rank-discounted utilitarianism in an appealing manner. In particular, the RDCLU SWO offers an escape from the Repugnant and Very Sadistic Conclusions.

However, practical application of the rank-discounted utilitarian criterion also requires explicit treatment of risk. This is the topic of Asheim and Zuber (2014).

Appendix: Proofs

We first prove the representation result by showing that statements (1) and (2) of Theorem 1 are equivalent. It is straightforward to show that an RDCLU SWO satisfies Axioms 1–7, so that statement (2) implies statement (1). Hence, to prove Theorem 1 we need to show that statement (1) implies statement (2); that is, that an SWR $\succsim$ on $\mathbf{X}$ satisfying Axioms 1–7 is an RDCLU SWO. This is shown by means of Lemmas 1–4.

We define the restriction $\succsim_n$ of $\succsim$ to $\mathbb{R}^n$ in the following way: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \succsim_n \mathbf{y}$ if and only if $\mathbf{x} \succsim \mathbf{y}$. We begin by establishing a representation result for sets of allocations with the same finite population size.

Lemma 1 *If the SWR $\succsim$ satisfies Axioms 1–5, then there exist $\beta \in \mathbb{R}_{++}$ and a continuous increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\mathbf{x} \succsim_n \mathbf{y} \iff \sum\nolimits_{r=1}^{n} \beta^r u(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} \beta^r u(y_{[r]}) \,. \tag{A.1}$$

Proof. For any $n \geq 3$, we show that the relation $\succsim_n$ satisfies the following properties proposed by Ebert (1988): Continuous Order, Monotonicity, Symme-

try and Independence with Respect to Ordered Vectors. By Ebert (1988, Theorem 1), this implies that there exist continuous increasing functions u_r^n such that for all $n \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \succsim_n \mathbf{y} \Longleftrightarrow \sum\nolimits_{r=1}^{n} u_r^n(x_{[r]}) \;\; \geq \;\; \sum\nolimits_{r=1}^{n} u_r^n(y_{[r]})\,. \tag{A.2}$$

The fact that $\succsim_n$ is a continuous order follows from Axioms 1 and 2. The fact that it satisfies Monotonicity and Symmetry follows from Axiom 3. For the independence condition, we can apply Gorman's (1968) theorem on the ordered set $\{\mathbf{x} \in \mathbb{R}^n : x(1) \leq \ldots \leq x(n)\}$. By Axiom 4, we know that all sets $\{1, 2, \ldots, t\}$ for $1 < t < n$ are separable. By Axiom 5, we know that all sets $\{t, t+1, \ldots, n\}$ for $1 < t < n$ are separable. By intersections of such separable subsets, we can obtain any subsets $\{t, t{+}1\}$, $1 \leq t < n$, which are therefore separable by Gorman's (1968) theorem. By unions of such subsets we can obtain any subset of indices in $\{1, \ldots, n\}$ so that they are also separable by Gorman's (1968) theorem. This corresponds to Ebert's (1988) Independence with Respect to Ordered Vectors. We may set $u_r^n(0) = 0$ for all $r \leq n$; this normalization can be performed without loss of generality because it leaves the ranking unchanged.

Now, representation (A.2) exists for $\succsim_n$ whatever $n \in \mathbb{N}$. Furthermore, by Axiom 4 we have the following equivalences (for $z \geq \{x_{[n]}, y_{[n]}\}$):

$$\begin{aligned}\sum\nolimits_{r=1}^{n} u_r^n(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} u_r^n(y_{[r]}) \;\; &\Longleftrightarrow \;\; \mathbf{x} \succsim_n \mathbf{y} \;\; \Longleftrightarrow \;\; (\mathbf{x}, z) \succsim_{n+1} (\mathbf{y}, z) \\ &\Longleftrightarrow \;\; \sum\nolimits_{r=1}^{n} u_r^{n+1}(x_{[r]}) + u_{n+1}^{n+1}(z) \geq \sum\nolimits_{r=1}^{n} u_r^{n+1}(y_{[r]}) + u_{n+1}^{n+1}(z)\,.\end{aligned}$$

By standard uniqueness results for additive functions on rank-ordered sets, we can take (after the appropriate normalization) $u_r^n \equiv u_r^{n+1}$. We can henceforth drop the superscript n in functions u_r^n.

By Axiom 5 we have the following equivalences (for $z \leq \min\{x_{[n]}, y_{[n]}\}$):

$$\sum\nolimits_{r=1}^{n} u_r(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} u_r(y_{[r]}) \;\; \Longleftrightarrow \;\; \mathbf{x} \succsim_n \mathbf{y} \;\; \Longleftrightarrow \;\; (\mathbf{x}, z) \succsim_{n+1} (\mathbf{y}, z)$$

$$\iff \sum_{r=1}^{n} u_{r+1}(x_{[r]}) + u_1(z) \geq \sum_{r=1}^{n} u_{r+1}(y_{[r]}) + u_1(z)\,.$$

By the cardinality of the additive representation and the normalization condition, there must exists a $\beta > 0$ such that $u_{r+1}(y) = \beta u_r(y)$ for any $y \in \mathbb{R}$. Note that β does not depend on r. We obtain (A.1), where $u \equiv u_1$ is a continuous increasing function from $\mathbb{R}$ to $\mathbb{R}$.

By Axiom 4 the proof can be extended to the case where $n = 1$ or $n = 2$. ■

Let $c \in \mathbb{R}_+$ be the critical-level parameter introduced in Axiom 6. Define $\mathbf{X}_c$ as the non-empty set $\{\mathbf{x} \in \mathbf{X} : \mathbf{x}_{[]} \leq (c)_{n(\mathbf{x})}\}$ of finite allocations where wellbeing does not exceed c. We obtain a representation result for finite allocations (with variable population) where wellbeing does not exceed c.

Lemma 2 *If the SWR $\succsim$ satisfies Axioms 1–6 and $c \in \mathbb{R}_+$ is the critical-level parameter of Axiom 6, then there exist $\beta \in \mathbb{R}_{++}$ and a continuous increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$,*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right)\,.$$

Proof. Assume that $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$ and $n(\mathbf{x}) \leq n(\mathbf{y})$ and let $k = n(\mathbf{y}) - n(\mathbf{x})$. Then:

$$\begin{aligned}
\mathbf{x} \succsim \mathbf{y} &\iff \left(\mathbf{x}, (c)_k\right) \succsim_{n(\mathbf{y})} \mathbf{y} \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r u(x_{[r]}) + \sum_{r=n(\mathbf{x})+1}^{n(\mathbf{y})} \beta^r u(c) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r u(y_{[r]}) \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r u(x_{[r]}) + \sum_{r=n(\mathbf{x})+1}^{n(\mathbf{y})} \beta^r u(c) - \sum_{r=1}^{n(\mathbf{y})} \beta^r u(c) \\
&\qquad\qquad \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r u(y_{[r]}) - \sum_{r=1}^{n(\mathbf{y})} \beta^r u(c) \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right)
\end{aligned}$$

since $\mathbf{x} \sim \left(\mathbf{x}, (c)_k\right)$ by k applications of Axiom 6, using Axiom 5 repeatedly to ensure that the allocation is in $\mathbb{R}^n$ when Axiom 6 is applied. ■

The following lemma shows that adding Axiom 7 implies that the rank utility discount factor, β, is smaller than 1.

Lemma 3 *If the SWR $\succsim$ satisfies Axioms 1–7 and $c \in \mathbb{R}_+$ is the critical-level parameter of Axiom 6, then there exist $0 < \beta < 1$ and a continuous increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$,*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum\nolimits_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum\nolimits_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) .$$

Proof. Let x, $y \in \mathbb{R}$ satisfy that $c \geq x > y$ and let $k \in \mathbb{N}$. Assume that there exist $z \in \mathbb{R}$ and $\ell \in \mathbb{N}$ such that $(x)_k \succ (z)_\ell \succ (y)_k$ (note that $z < c$, because otherwise, by Axioms 1, 3 and 6, $(z)_\ell \succsim (c)_\ell \sim (c)_k \succsim (x)_k$, a contradiction). By Lemma 2, this means that (for $\beta \neq 1$; the case $\beta = 1$ can be treated similarly):

$$\tfrac{\beta^k - 1}{\beta - 1}\big(u(x) - u(c)\big) > \tfrac{\beta^\ell - 1}{\beta - 1}\big(u(z) - u(c)\big) > \tfrac{\beta^k - 1}{\beta - 1}\big(u(y) - u(c)\big) .$$

When $\beta > 1$, $\lim_{\ell \to \infty}(\beta^\ell - 1)/(\beta - 1) = \infty$ so that

$$\lim_{\ell \to \infty} \tfrac{\beta^\ell - 1}{\beta - 1}\big(u(z) - u(c)\big) = -\infty .$$

Hence there exists $N > \ell$ such that

$$\tfrac{\beta^n - 1}{\beta - 1}\big(u(z) - u(c)\big) < \tfrac{\beta^{n(\mathbf{x})} - 1}{\beta - 1}\big(u(y) - u(c)\big)$$

for all $n \geq N$, a contradiction of Axiom 7. ■

Finally, we extend the representation to the entire domain $\mathbf{X}$ of all finite allocations by showing that any finite allocation $\mathbf{x}$ can be made as bad as an allocation where all individuals are at the critical level c by adding sufficiently many people at a low wellbeing level z, and thus indifferent to an egalitarian allocation where each individual's wellbeing equals $x \leq c$. This allows us to combine the representations of Lemmas 1 and 2, and shows that statement (1) of Theorem 1 implies statement (2).

Lemma 4 *If the SWR $\succsim$ satisfies Axioms 1–7, then there exist $0 < \beta < 1$ and a continuous increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum\nolimits_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum\nolimits_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) .$$

Proof. *Step 1: For any $n \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$ satisfying $z \leq x_{[1]}$ and $z < c$, there exists $k(\mathbf{x})$ such that for all $k \geq k(\mathbf{x})$, $(\mathbf{x}, (z)_k) \prec (c)_{n+k}$.*

By Lemma 3, we know that $\beta < 1$ in the representation on $\mathbf{X}_c$. By Lemma 1, the property extends to $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

For any $\mathbf{x} \in \mathbb{R}^n$, the n-equally distributed equivalent of $\mathbf{x}$ denoted $e_n(\mathbf{x})$ is the real number $x \in \mathbb{R}$ such that $(x)_n \sim_n \mathbf{x}$. Axioms 1–3 imply that $e_n : \mathbb{R}^n \to \mathbb{R}$ is well-defined. By Lemma 1 and since Axioms 1–5 hold, it is defined as follows:

$$e_n(\mathbf{x}) = u^{-1}\Big(\tfrac{1-\beta}{1-\beta^n}\sum\nolimits_{r=1}^{n}\beta^{r-1}u(x_{[r]})\Big)\,.$$

Now let $z \leq x_{[1]}$ and $z < c$. We obtain the following expression:

$$\begin{aligned} e_{n+k}\big(\mathbf{x}, (z)_k\big) &= u^{-1}\Big(\tfrac{1-\beta}{1-\beta^{n+k}}\Big(\sum\nolimits_{r=1}^{k}\beta^{r-1}u(z) + \sum\nolimits_{r=k+1}^{n+k}\beta^{r-1}u(x_{[r]})\Big)\Big) \\ &= u^{-1}\Big(\tfrac{1-\beta^k}{1-\beta^{n+k}}u(z) + \tfrac{\beta^k-\beta^{n+k}}{1-\beta^{n+k}}\cdot\tfrac{1-\beta}{1-\beta^n}\sum\nolimits_{r=1}^{n}\beta^{r-1}u(x_{[r]})\Big)\,. \end{aligned}$$

If $x_{[n]} \leq c$, then $e_{n+k}\big(\mathbf{x}, (z)_k\big) \leq c$ for all $k \in \mathbb{N}$ and Step 1 is completed. Therefore, assume $x_{[n]} > c$ which, since $z \leq x_{[1]}$ and $z < c$, implies that $z < e_n(\mathbf{x})$.

Write $a_k := (1-\beta^k)/(1-\beta^{n+k})$; because $0 < \beta < 1$, $(a_k)_{k\in\mathbb{N}}$ is an increasing sequence converging to 1. Since $z < e_n(\mathbf{x})$ and

$$e_{n+k}\big(\mathbf{x}, (z)_k\big) = u^{-1}\big(a_k u(z) + (1-a_k)u\big(e_n(\mathbf{x})\big)\big)\,,$$

it follows that $e_{n+k+1}\big(\mathbf{x}, (z)_{k+1}\big) < e_{n+k}\big(\mathbf{x}, (z)_k\big)$ and $e_{n+k}\big(\mathbf{x}, (z)_k\big)$ tends to z when k tends to infinity. As $z < c$, we deduce that, for any $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^n$, there exists $k(\mathbf{x}) \in \mathbb{N}$ such that, for any $k \geq k(\mathbf{x})$, $e_{n+k}\big(\mathbf{x}, (z)_k\big) < c$.

Step 2: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, choose z with $z \leq \min\{x_{[1]}, y_{[1]}\}$ and $z < c$, $\ell = \max\{k(\mathbf{x}), k(\mathbf{y})\}$, $x = e_{n(\mathbf{x})+\ell}\big(\mathbf{x}, (z)_\ell\big)$ and $y = e_{n(\mathbf{y})+\ell}\big(\mathbf{y}, (z)_\ell\big)$, and use $(\mathbf{x}, (z)_\ell) \sim (x)_{n(\mathbf{x})+\ell}$, $(\mathbf{y}, (z)_\ell) \sim (y)_{n(\mathbf{y})+\ell}$ and $(x)_{n(\mathbf{x})+\ell}$, $(y)_{n(\mathbf{y})+\ell} \in \mathbf{X}_c$ to establish the result.

Using the above definitions of x and y, we obtain the following equivalences by repeated applications of Axiom 5 and Lemma 2:

$$\mathbf{x} \succsim \mathbf{y} \iff (x)_{n(\mathbf{x})+\ell} \sim (\mathbf{x}, (z)_\ell) \succsim (\mathbf{y}, (z)_\ell) \sim (y)_{n(\mathbf{y})+\ell}$$

$$\iff \sum_{r=1}^{n(\mathbf{x})+\ell} \beta^r \left(u(x) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})+\ell} \beta^r \left(u(y) - u(c)\right) .$$

However, by Lemma 1,

$$\sum_{r=1}^{n(\mathbf{x})+\ell} \beta^r u(x) = \sum_{r=1}^{\ell} \beta^r u(z) + \beta^\ell \sum_{r=1}^{n(\mathbf{x})} \beta^r u(x_{[r]}) ,$$
$$\sum_{r=1}^{n(\mathbf{y})+\ell} \beta^r u(y) = \sum_{r=1}^{\ell} \beta^r u(z) + \beta^\ell \sum_{r=1}^{n(\mathbf{y})} \beta^r u(y_{[r]}) ,$$

using the fact $(x)_{n(\mathbf{x})+\ell} \sim (\mathbf{x}, (z)_\ell)$ and $(y)_{n(\mathbf{y})+\ell} \sim (\mathbf{y}, (z)_\ell)$. We obtain that

$$\sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right)$$

if and only if $\mathbf{x} \succsim \mathbf{y}$ by combining these result and rearranging terms. ■

Next, we provide a proof of Theorem 2.

Proof of Theorem 2. Assume that $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and u is a continuous and increasing function.

Part (i): This part follows directly from the observation that, on the domain $(0, 1]$, $\sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right)$ is a continuous function of β.

Part (ii): Since $\succsim_c^L$ is complete, it is sufficient to show that $\mathbf{x} \sim_c^L \mathbf{y}$ implies the existence of $\underline{\beta} \in (0, 1)$ such that $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0, \underline{\beta})$, and that $\mathbf{x} \succ_c^L \mathbf{y}$ implies the existence of $\underline{\beta} \in (0, 1)$ such that $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0, \underline{\beta})$.

$\mathbf{x} \sim_c^L \mathbf{y}$ *implies the existence of* $\underline{\beta} \in (0, 1)$ *such that* $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ *for all* $\beta \in (0, \underline{\beta})$. Let $n(\mathbf{x}) \geq n(\mathbf{y})$. By Definition 5, $(x_{[1]}, \ldots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{x})+1]}, \ldots, x_{[n(\mathbf{y})]}) = (c)_{n(\mathbf{x})-n(\mathbf{y})}$. By Definition 1, $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0, 1)$.

$\mathbf{x} \succ_c^L \mathbf{y}$ *implies the existence of* $\underline{\beta} \in (0, 1)$ *such that* $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ *for all* $\beta \in (0, \underline{\beta})$. Let $x = \min\{x_{[1]}, c\}$ and $y = \max\{y_{[n(\mathbf{y})]}, c\}$. Note that $x \leq y$. If $x = y$ and $\mathbf{x} \succ_c^L \mathbf{y}$, then by Definitions 1 and 5, $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0, 1)$. Hence, only the case where $x < y$ remains. By Definition 5, there are three cases. Case 1: there exists $R \in \{1, \ldots, n\}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > y_{[R]}$, where $n := \min\{n(\mathbf{x}), n(\mathbf{y})\}$. In this case, let $x' = x_{[R]}$ and $y' = y_{[R]}$. Case 2:

$n(\mathbf{x}) > n(\mathbf{y})$, $(x_{[1]}, \ldots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{y})+1]}, \ldots, x_{[n(\mathbf{x})]}) > (c)_{n(\mathbf{x})-n(\mathbf{y})}$. In this cas e, let $x' = x_{[R]}$ and $y' = c$, where $R := \min\{r > n(\mathbf{y}) : x_{[r]} > c\}$. Case 3: $n(\mathbf{x}) < n(\mathbf{y})$, $\mathbf{x}_{[]} = (y_{[1]}, \ldots, y_{[n(\mathbf{x})]})$ and $c > y_{[n(\mathbf{x})+1]}$. In this case, let $x' = c$ and $y' = y_{[n(\mathbf{x})+1]}$. Note that in all three cases, $x' > y'$. Define $\underline{\beta}$ by

$$(1-\underline{\beta})u(x') + \underline{\beta}u(x) = (1-\underline{\beta})u(y') + \underline{\beta}u(y)\,.$$

Then, by applying the following affine transformation:

$$\tfrac{1-\beta}{\beta}\Big(\sum\nolimits_{r=1}^{n(\mathbf{x})}\beta^r\left(u(x_{[r]}) - u(c)\right)\Big) + u(c) = (1-\beta)\sum\nolimits_{r=1}^{n(\mathbf{x})}\beta^{r-1}u(x_{[r]}) + \beta^{n(\mathbf{x})}u(c)\,,$$

it follows from Definition 1 that $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0, \underline{\beta})$. ■

Finally, we provide proofs of Propositions 1 and 2.

Proof of Proposition 1. *Part (i). Consider an NDCLU SWO* $\succsim$ *such that* $c > 0$. Assume that $\succsim$ satisfies the Weak Non-Sadism Condition: there exist $y \in \mathbb{R}_{--}$ and $k \in \mathbb{N}$ such that, for all $\mathbf{x} \in \mathbf{X}$, $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$, $\big(\mathbf{x}, (y)_k\big) \precsim \big(\mathbf{x}, (z)_n\big)$. Consider any $v > c > z > 0$. Choose $\ell \in \mathbb{N}$ such that[6]

$$f(\ell+k)\left(\tfrac{\ell}{\ell+k}\big(u(v)-u(c)\big) + \tfrac{k}{\ell+k}\big(u(y)-u(c)\big)\right) > 0. \tag{A.3}$$

It also possible to find $n \in \mathbb{N}$ such that[7]

$$f(\ell+n)\left(\tfrac{\ell}{\ell+n}\big(u(v)-u(c)\big) + \tfrac{n}{\ell+n}\big(u(z)-u(c)\big)\right) < 0. \tag{A.4}$$

By Definition 4, eqs. (A.3) and (A.4) imply that $\big((v)_\ell, (y)_k\big) \succ \big((v)_\ell, (z)_n\big)$, which is a contradiction.

Consider an NDCLU SWO $\succsim$ *such that* $c = 0$. If the function f is not bounded above, then for all $y > z > 0$ and $k \in \mathbb{N}$, there is $n > k$ such that

[6]There necessarily exists such an $\ell \in \mathbb{N}$ because $f(\ell+k) > 0$ and $(\ell/(\ell+k))\big(u(v)-u(c)\big) + (k/(\ell+k))\big(u(y)-u(c)\big)$ is arbitrarily close to $u(v)-u(c) > 0$ for large values of ℓ.

[7]There necessarily exists such an $n \in \mathbb{N}$ because $f(\ell+n) > 0$ and $(\ell/(\ell+n))\big(u(v)-u(c)\big) + (n/(\ell+n))\big(u(z)-u(c)\big)$ is arbitrarily close to $u(z)-u(c) < 0$ for large values of n.

$f(k)\big(u(y)-u(0)\big) < f(n)\big(u(z)-u(0)\big)$. Hence, by Definition 4, $(y)_k \prec (z)_n$, which is the Repugnant Conclusion. Therefore, assume that the function f is bounded above. Assume furthermore that $\succsim$ satisfies the Weak Non-Sadism Condition: there exist $y \in \mathbb{R}_{--}$ and $k \in \mathbb{N}$ such that, for all $\mathbf{x} \in \mathbf{X}$, $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$, $\big(\mathbf{x}, (y)_k\big) \precsim \big(\mathbf{x}, (z)_n\big)$. For $v > 0$, choose like above $\ell \in \mathbb{N}$ such that:

$$f(\ell + k)\left(\tfrac{\ell}{\ell+k}\big(u(v) - u(0)\big) + \tfrac{k}{\ell+k}\big(u(y) - u(0)\big)\right) > 0. \tag{A.5}$$

Denote the left-hand-side by M. Because f is bounded above and u is continuous and increasing, there is $z > 0$ such that $f(n)\big(u(z) - u(0)\big) < M$ for all $n \in \mathbb{N}$. Hence, by the same reasoning as above, there exists $n \in \mathbb{N}$ such that,

$$f(\ell + n)\left(\tfrac{\ell}{\ell+n}\big(u(v) - u(0)\big) + \tfrac{n}{\ell+n}\big(u(z) - u(0)\big)\right) < M. \tag{A.6}$$

By Definition 4, eqs. (A.5) and (A.6) imply that $\big((v)_\ell, (y)_k\big) \succ \big((v)_\ell, (z)_n\big)$, which is a contradiction.

The arguments of Part (i) hold even if the function u is bounded above.

Part (ii). An RDCLU SWO $\succsim$ avoids the Weak Repugnant Conclusion. If $y > z > c$, then there exists $k \in \mathbb{N}$ determined by the requirement that

$$\tfrac{1-\beta^k}{1-\beta}\left(u(y) - u(c)\right) \geq \tfrac{1}{1-\beta}\left(u(z) - u(c)\right)\,, \tag{A.7}$$

implying that $(y)_k \succ (z)_n$ for all $n \in \mathbb{N}$ according to the RDCLU SWO (cf. Definition 1). This holds even if the function u is bounded above.

An RDCLU SWO $\succsim$ satisfies the Weak Non-Sadism Condition if the function u is bounded above. Let $\bar{u} := \sup\{u(x) : x \in \mathbb{R}\}$ and choose $y \in \mathbb{R}_{--}$ and $k \in \mathbb{N}$ such that

$$\tfrac{1-\beta^k}{1-\beta}\left(u(y) - u(0)\right) + \tfrac{\beta^k}{1-\beta}\left(\bar{u} - u(0)\right) \leq 0\,. \tag{A.8}$$

Then it follows from the fact that $\bar{u} \geq u(c) \geq u(0)$ that, for any initial allocation $\mathbf{x} \in \mathbf{X}$ where all individuals have positive lifetime wellbeing, $\big(\mathbf{x}, (y)_k\big) \prec \big(\mathbf{x}, (z)_n\big)$

for all $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$. This obtains since eq. (A.8) implies

$$\tfrac{1-\beta^k}{1-\beta}\left(u(y)-u(c)\right)+\tfrac{\beta^k}{1-\beta}\left(\bar{u}-u(c)\right) \leq \tfrac{1}{1-\beta}\left(u(0)-u(c)\right) ,$$

where the r.h.s. is smaller than the welfare of any allocation where all individuals have positive wellbeing. Including individuals in $\mathbf{x}$ with wellbeing at or below y does not change the ranking, while including individuals in $\mathbf{x}$ at y and then raise their wellbeing towards 0 contributes less to $\big(\mathbf{x},(y)_k\big)$ than to $\big(\mathbf{x},(z)_n\big)$ since their rank is higher in the former allocation than in the latter. Hence, for any initial allocation $\mathbf{x} \in \mathbf{X}$, $\big(\mathbf{x},(y)_k\big) \precsim \big(\mathbf{x},(z)_n\big)$, showing that an RDCLU SWO satisfies the Weak Non-Sadism Condition if the function u is bounded above. ■

Proof of Proposition 2. *Part (i). Consider an NDCLU SWO* $\succsim$ *such that the function* f *is bounded above.* If, for some $n \in \mathbb{N}$, $f(n) > f(n+1)$, then Definition 4 implies that for, any $z \in \mathbb{R}_{--}$, $(z)_n \prec (z)_{n+1}$, so that the Negative Mere Addition Principle is not satisfied. Therefore, assume that f is non-decreasing. Let $y < z < 0$. If there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\big((y)_n,(z)_k\big) \succ (y)_n$, then by completeness and transitivity of an NDCLU SWO, there must exist $0 \leq \ell < k$ such that $\big((z)_n,(y)_{\ell+1}\big) \succ \big((z)_n,(y)_\ell\big)$, a violation of the Negative Mere Addition Principle. Hence, to establish a violation of the Negative Mere Addition Principle it is sufficient to show that there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\big((y)_n,(z)_k\big) \succ (y)_n$ or equivalently, by Definition 4,

$$\frac{f(n)}{f(n+k)} > \frac{\frac{n}{n+k}\big(u(c)-u(y)\big)+\frac{k}{n+k}\big(u(c)-u(z)\big)}{u(c)-u(y)} . \tag{A.9}$$

For sufficiently small $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that

$$\frac{f(n)}{f(n+k)} \geq \frac{u(c)-u(z)}{u(c)-u(y)}+\varepsilon$$

for all $k \in \mathbb{N}$, since any non-decreasing bounded sequence converges and $u(y) < u(z) < u(c)$. The result follows since the numerator of the right-hand side of (A.9) is arbitrarily close to $u(c)-u(z)$ for large values of k.

Consider an NDCLU SWO $\succsim$ such that the function f is not bounded above. If $c > 0$, then, for all $z \in \mathbb{R}_{++}$ with $z < c$, $f(n)\big(u(z) - u(c)\big)$ tends to minus infinity when n goes to infinity. Therefore, by Definition 4, for any $\mathbf{x} \in \mathbf{X}$ there exists $n \in \mathbb{N}$ such that $\mathbf{x} \succ (z)_n$, which is the Very Sadistic Conclusion. If on the other hand $c = 0$, then, for all $z \in \mathbb{R}_{++}$, $f(n)u(z)$ tends to infinity when n goes to infinity. Hence, by Definition 4, for all $y \in \mathbb{R}_{++}$ with $y > z > 0$ and $k \in \mathbb{N}$, there is $n > k$ such that $(y)_k \prec (z)_n$, which is the Repugnant Conclusion.

Part (ii). An RDCLU SWO $\succsim$ avoids the Repugnant Conclusion. By Proposition 1(ii), any RDCLU SWO avoids the Weak Repugnant Conclusion and therefore avoids the Repugnant Conclusion.

An RDCLU SWO $\succsim$ avoids the Very Sadistic Conclusion. Choose $y \in \mathbb{R}_{--}$ and $k \in \mathbb{N}$ such that $(1-\beta^k)\big(u(y) - u(c)\big) \leq u(0) - u(c) \leq (1-\beta^n)\big(u(0) - u(c)\big)$ for all $n \in \mathbb{N}$. By Definition 1, $(y)_k \precsim (0)_n \prec (z)_n$ for all $z \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$. Thus an RDCLU SWO avoids the Very Sadistic Conclusion.

An RDCLU SWO $\succsim$ satisfies the Negative Mere Addition Principle. This follows since c is a lower bound for the context-dependent critical level for $\succsim$. ■

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