Two-stage Threshold Representations*

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Abstract

We study two-stage choice procedures in which the decision maker first preselects the alternatives whose values according to a criterion pass a menu-dependent threshold, and then maximizes a second criterion to narrow the selection further. This framework overlaps with several existing models that have various interpretations and impose various additional restrictions on behavior. We show that the general class of procedures is characterized by acyclicity of the revealed "first-stage separation relation."

1 Introduction

Several recent contributions to axiomatic choice theory have studied two-stage procedures in which first a subset of alternatives is preselected and then a maximization operation narrows the selection further. Examples include Cherepanov et al. [3], Lleras et al. [4], Manzini and Mariotti [5], Masatlioglu et al. [6], and Tyson [11]. The interpretations of these models vary considerably, with the preselection stage used in [3] to express the desire for a psychological "rationalization" of the eventual choice, in [4] to allow active consideration of a subset of alternatives only, and in [5] to capture a "noncompensatory heuristic."

In this paper we investigate two-stage procedures in which the preselection mechanism has a threshold representation of the sort considered by Aleskerov and Monjardet [1] and Tyson [10]. Such a representation involves a criterion function f on the set of alternatives and a threshold function θ on the set of menus. Alternative x is preselected from menu A if its value $f(x)$ is at least the threshold $\theta(A)$ assigned to this choice problem. Writing g for the second-stage maximand, the solutions of the constrained optimization problem

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$$\max_{x \in A} g(x) \text{ subject to } f(x) \geq \theta(A)$$

are then the options selected from menu A by the full two-stage procedure.

This type of "two-stage threshold" (TST) representation is of interest because it overlaps with a number of the theories of choice mentioned above. Cognitive mechanisms for dealing with complex decision problems, such as attention and satisficing, are naturally modeled with thresholds. And since the resulting preselection may be coarse, with numerous alternatives achieving the threshold, a second criterion can help to further refine the options.

Our goal is to determine how the TST representation constrains behavior independently of any extra restrictions implied by more specific models. This is accomplished by our main result, which characterizes the representation in terms of a single axiom on the choice function. The axiom imposes acyclicity on the "first-stage separation relation" encoding when one alternative is chosen over another despite evidence that they cannot be distinguished at the second stage. This is of course implied by the acyclicity condition that characterizes the one-stage threshold model (a result included below for the sake of comparison).

In general the consequences of adding a second stage to a choice-theoretic model can be difficult to predict. Allowing an ordinary preference maximizer to break his indifference by maximizing a second criterion does not change the behavioral possibilities.[1] In contrast, the TST model turns out to have considerably less empirical content than its one-stage counterpart. This is shown most clearly by a corollary of our main result stating that for the special case of single-valued choice functions, the TST representation places *no constraints whatsoever* on behavior. We conclude that models consistent with the TST framework get most of their logical strength not from the representation itself, but rather from the additional restrictions they impose.

We characterize TST representations in Section 2, discuss more specialized models in Section 3, and prove our main result in Section 4.

2 Characterization results

Fix a nonempty, finite set X, and let $\mathcal{D} \subseteq \mathcal{A} = 2^X \setminus \{\emptyset\}$. Each $x \in X$ is an *alternative*, and each $A \in \mathcal{D}$ is a *menu*. A *choice function* is any $C : \mathcal{D} \to \mathcal{A}$ such that $\forall A \in \mathcal{D}$ we have $C(A) \subseteq A$. Here $C(A)$ is the *choice set* assigned to A, with the interpretation that those and only those alternatives in $C(A)$ could be chosen from this menu. Without loss of generality, assume that $\forall x \in X$ we have $\{x\} \in \mathcal{D}$.

We study the class of choice functions that select alternatives from menu A by maximizing $g(x)$ subject to $f(x) \geq \theta(A)$, where $f, g : X \to \Re$ and $\theta : \mathcal{D} \to \Re$. In the context of such a representation we refer to f as the *primary criterion*, g as the *secondary criterion*, and θ as the *threshold map*. The triple $\langle f, \theta, g \rangle$ will be called a *profile*.

We define first the *threshold set* containing those alternatives on a menu with sufficiently high values of the primary criterion.

Definition 1. Given a pair $\langle f, \theta \rangle$ and an $A \in \mathcal{D}$, let $\Gamma(A|f, \theta) = \{x \in A : f(x) \geq \theta(A)\}$.

[1] Lexicographic maximization of two weak orders is behaviorally equivalent to maximization of a single weak order, both being characterized by Richter's [7, p. 637] congruence axiom.

The model under investigation can now be defined formally.

Definition 2. A *two-stage threshold representation* of C is a profile $\langle f, \theta, g \rangle$ such that $\forall A \in \mathcal{D}$ we have $C(A) = \operatorname{argmax}_{x \in \Gamma(A|f,\theta)} g(x)$.

For the sake of concreteness, we provide an illustration of how the functions f, θ, and g interact to determine choice behavior, using a multiplicative notation for enumerated sets.

Example 1. Let $f(x) = 1$, $f(y) = 0$, $f(z) = 2$, $g(x) = 1$, $g(y) = 1$, $g(z) = 0$, $\theta(xy) = 1$, $\theta(xz) = 2$, $\theta(yz) = 0$, and $\theta(xyz) = 0$. The profile $\langle f, \theta, g \rangle$ is then a TST representation of the choice function given by $C(xy) = x$, $C(xz) = z$, $C(yz) = y$, and $C(xyz) = xy$.

Among other things, this demonstrates that the TST model can accommodate cyclical binary choices. However, a slight modification to the choice function in this example suffices to show that not all varieties of behavior are allowed.

Example 2. Let $C(xy) = x$, $C(xz) = z$, $C(yz) = y$, and $C(xyz) = xyz$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $C(xyz) = xyz$ would imply $g(x) = g(y) = g(z)$. But then the remaining choice data would imply $f(x) \geq \theta(xy) > f(y) \geq \theta(yz) > f(z) \geq \theta(xz) > f(x)$, a contradiction.

Determining the empirical content of the two-stage threshold model requires us to identify conditions that distinguish choice functions consistent with a TST representation from those that are not. To do this we shall employ a number of binary relations that are "behavioral" in the sense of being derived from C, beginning with the *separation relation.*

Definition 3. Let xSy if $\exists A \in \mathcal{D}$ such that $x \in C(A)$ and $y \in A \setminus C(A)$.

In other words, x is separated from y when there exists a menu on which both are available, x is choosable, and y is not. If choices maximize a utility function, then separation reveals the corresponding strict preferences. On the other hand, indifference is revealed by choosability from the same menu, encoded in the *togetherness relation.*

Definition 4. Let xTy if $\exists A \in \mathcal{D}$ such that $x, y \in C(A)$.

It is also useful to define the transitive closure of T, which we shall refer to as the *extended togetherness relation.*

Definition 5. Let xEy if $\exists z_1, z_2, \ldots, z_n \in X$ such that $x = z_1 T z_2 T \cdots T z_n = y$.

Note that since T is both reflexive and symmetric, E is an equivalence.[2] In classical revealed preference analysis, E-equivalence classes amount to revealed indifference curves.

When C has a TST representation $\langle f, \theta, g \rangle$ instead of an ordinary utility representation, the relations S and T must be interpreted differently. Here xSy implies either $f(x) > f(y)$ or $g(x) > g(y)$, since the separation of x from y must occur — speaking in terms of the representation — at either the first or the second stage. Meanwhile, xTy tells us nothing about the first stage but ensures that $g(x) = g(y)$, and likewise for extended togetherness.

[2] A relation R on X is an equivalence if it is reflexive ($\forall x \in X$ we have xRx), symmetric ($\forall x, y \in X$ we have xRy only if yRx), and transitive ($\forall x, y, z \in X$ we have $xRyRz$ only if xRz).

Though neither S nor T by itself says anything definitive about the first stage of a TST representation, they can be used together to elicit such information. Indeed, we have seen this already in Example 2, where xTy implied $g(x) = g(y)$, and so xSy could only mean that $f(x) > f(y)$. This remains true for alternatives related by extended togetherness, which is to say that separations between alternatives in the same E-equivalence class must be attributed to the first stage. To capture this reasoning, we define the *first-stage separation relation.*

Definition 6. Let xFy if both xEy and xSy.

This definition suggests a necessary condition for the TST model. Example 2 shows how $xFyFzFx$ leads to a contradiction, and an F-cycle of any length would yield the same result. The condition is thus that the first-stage separation relation be acyclic; i.e., that there be no S-cycle within an E-equivalence class.[3] Note that this is satisfied in Example 1, where E partitions the alternatives as $\{xy, z\}$ and the S-cycles present are $xSySzSx$ and $xSzSx$.

Remarkably, acyclicity of F turns out also to be sufficient for the TST framework.

Theorem. *A choice function has a two-stage threshold representation if and only if the relation F is acyclic.*

This result involves no monotonicity (e.g., contraction or expansion consistency) conditions or congruence axioms of the sort common in the revealed preference literature. Likewise, no constraint links pairwise choices to those from larger menus, even if pairwise choice data happen to be available. The single condition needed is straightforward to state, and its role can be appreciated in contexts as simple as Examples 1 and 2.

The above theorem is proved formally in Section 4; here we merely sketch the argument for sufficiency. Given acyclicity of F, we first construct a secondary criterion that is constant on E-equivalence classes and otherwise orders the alternatives arbitrarily. We then construct a primary criterion that orders the alternatives in agreement with F inside E-equivalence classes — the acyclicity condition ensuring that no contradiction arises at this point — and otherwise *in opposition to* the secondary criterion. The threshold for each menu A is set equal to the minimum of the primary criterion over the choice set $C(A)$. And it can then be confirmed that the resulting profile is a TST representation of the choice function.[4]

Two special cases are worth mentioning. First, when the secondary criterion is constant, the second stage vanishes and for each menu the choice and threshold sets coincide.

Definition 7. A *one-stage threshold representation* of C is a pair $\langle f, \theta \rangle$ such that $\forall A \in \mathcal{D}$ we have $C(A) = \Gamma(A|f, \theta)$.

Under such a representation any separation xSy implies $f(x) > f(y)$, so clearly the entire relation S must be acyclic. Again this necessary condition can be shown also to be sufficient, yielding a characterization obtained by Aleskerov and Monjardet [1].

[3] A relation R on X is acyclic if $\forall x_1, x_2, \ldots, x_n \in X$ we have $x_1 R x_2 R \cdots R x_n$ only if $\neg[x_n R x_1]$.

[4] For instance, take the choice function in Example 1. Here since xEy we need $g(x) = g(y)$, and we can arbitrarily set $g(z) < g(x)$. Since xFy we need $f(x) > f(y)$, and since $g(z) < g(x)$ we also need $f(z) > f(x)$. Finally, the thresholds will satisfy $\theta(xy) = f(x)$, $\theta(xz) = f(z)$, $\theta(yz) = f(y)$, and $\theta(xyz) = f(y)$. Note that this constructed profile will not be the same in all respects — even ordinally — as the original profile in Example 1, but will nevertheless be a TST representation of the choice function.

Proposition. *A choice function has a one-stage threshold representation if and only if the relation S is acyclic.*

The second special case is that of single-valued choice functions. When all choice sets are singletons the relations T and E are both empty, and hence F too is empty. But then F is trivially acyclic, allowing us to conclude the following as a consequence of our theorem.

Corollary. *Any single-valued choice function has a two-stage threshold representation.*

In this context the sufficiency argument outlined above is much simplified. Since E is empty, we can take g to be an arbitrary one-to-one function. Furthermore, since F is empty we can set $f = -g$. A menu's threshold will of course be the f-value of the unique element of the choice set. With the two criteria one-to-one and diametrically opposed, it is then immediate that the profile constructed makes up a TST representation.

3 More specialized models

To the best of our knowledge two-stage threshold representations have not previously been studied in isolation. However, several authors have proposed theories that overlap with the TST model, based on a variety of hypotheses about the process of decision making.

1. Lleras et al. [4] introduce a model in which the alternatives actively considered by the decision maker are a subset of those available. To obtain behavioral restrictions, they require that for any two menus A and B such that $A \subseteq B$, an alternative $x \in A$ is considered in choice problem B only if it is also considered in problem A. The TST framework will generate a special case of this model if the primary criterion f measures the propensity of an alternative to be considered, the threshold map θ returns minimum f-values for consideration, and θ is monotone (i.e., $A \subseteq B$ implies $\theta(A) \leq \theta(B)$).[5] The secondary criterion g here represents an ordinary utility function.

2. Masatlioglu et al. [6] suppose that alternatives are pre-selected not by active consideration but rather by the decision maker's awareness of them. Here it is assumed that if all alternatives perceived in choice problem B are available on some menu $A \subseteq B$, then the options perceived in problems A and B will be identical.[6] Once again a special case of this model can be generated by the TST structure: With f and θ governing awareness, and g measuring utility, a sufficient condition for the above assumption is that $A \subseteq B$ and $\max f[B \setminus A] < \theta(B)$ together imply $\theta(A) = \theta(B)$.

3. Tyson [11] studies a model in which the decision maker's preferences among alternatives are perceived imperfectly, the coarseness of this perception is increasing with respect to $\subseteq$, and the choice between perceived-preference-maximal options is controlled by a

[5] Indeed, many of the illustrations provided by Lleras et al., such as considering "the n cheapest options," are consistent with the TST special case. Related models are described by Salant and Rubinstein [8] under the rubric of "choice with frames."

[6] In stating this "attention filter" assumption Masatlioglu et al. let $A = B \setminus \{x\}$. But this is without loss of generality when $\mathcal{D} = \mathcal{A}$, as imposed in [6].

binary relation that can be interpreted as a measure of relative "salience." This model admits a TST representation in which f is the utility function, θ returns satisfaction levels, and g is a salience mapping. In addition, the model imposes the "expansiveness" restriction that $A \subseteq B$ and $\max f[A] \geq \theta(B)$ together imply $\theta(A) \geq \theta(B)$.

These theories interpret the components of the profile $\langle f, \theta, g \rangle$ in quite different ways. In particular, the first two models view the secondary criterion as the appropriate welfare measure, while the third assigns this role to the primary criterion. Moreover, in the consideration and awareness frameworks the threshold map controls whether or not alternatives advance to the utility-maximization stage.[7] This contrasts with the third framework, where θ interacts directly with the utility function and implements a form of satisficing behavior.

Interpretation aside, all three of the above models impose restrictions beyond the basic two-stage threshold structure that constrain C in various ways. Since our theorem identifies the empirical content of the TST structure itself, the incremental content of these additional restrictions amounts to the logical gap between our F-acyclicity condition and the axioms that characterize the more elaborate models. For instance, in Tyson [11] the combination of "Weak Congruence" and "Base Transitivity" is stronger than acyclicity of F, and the extra logical force is what yields the expansiveness property of the TST representation.

Of course, our corollary establishes that the TST framework has no intrinsic empirical content when C is single-valued. Under this assumption the axioms that characterize a more specialized model use all of their logical force to impose restrictions on the representation. For example, the TST special case of the consideration model in Lleras et al. [4] lacks empirical content in the single-valued setting until we require the threshold map to be monotone.

4 Proof of Theorem

Let C have a TST representation $\langle f, \theta, g \rangle$. If $\exists x_1, x_2, \ldots, x_n \in X$ such that $x_1 F x_2 F \cdots F x_n$, then $x_1 E x_2 E \cdots E x_n$ and so $g(x_1) = g(x_2) = \cdots = g(x_n)$. We have also $x_1 S x_2 S \cdots S x_n$, and it follows that $f(x_1) > f(x_2) > \cdots > f(x_n)$. Since both $f(x_n) \leq f(x_1)$ and $g(x_n) \leq g(x_1)$, we have $\neg[x_n S x_1]$ and so $\neg[x_n F x_1]$. Hence F is acyclic.

Conversely, suppose that F is acyclic. Write $K(x)$ for the E-equivalence class of $x \in X$, define $\mathcal{K} = \{K(x) : x \in X\}$, and let $\gg$ be any linear order on $\mathcal{K}$.[8] Let $\phi : \mathcal{K} \to \Re$ be any representation of $\gg$, and for each $x \in X$ assign $g(x) = -\phi(K(x))$. The function $g : X \to \Re$ so defined will be the secondary criterion. Observe that xEy only if $K(x) = K(y)$ and hence $g(x) = g(y)$. Finally, let xQy if either xFy or $g(x) < g(y)$, and note that then we have xQy only if $g(x) \leq g(y)$.

Lemma. *The relation Q is acyclic.*

Proof. Suppose that $\exists x_1, x_2, \ldots, x_n \in X$ such that $x_1 Q x_2 Q \cdots Q x_n$, and define $x_{n+1} = x_1$. If $x_n Q x_1$, then since F is acyclic there exists a $k \leq n$ such that $g(x_k) < g(x_{k+1})$. But since

[7] Note that using thresholds to model phenomena related to attention and awareness is natural in light of how the human visual, auditory, and other sensory systems operate (see, e.g., Anderson [2]).

[8] A relation R on X is a linear order if it is asymmetric ($\forall x, y \in X$ we have xRy only if $\neg[yRx]$), negatively transitive ($\forall x, y, z \in X$ we have xRz only if either xRy or yRz), and weakly complete ($\forall x, y \in X$ we have $x \neq y$ only if either xRy or yRx).

$x_{k+1}Qx_{k+2}Q\cdots Qx_nQx_1Qx_2Q\cdots Qx_k$, we have also $g(x_{k+1}) \leq g(x_k)$, a contradiction. Hence $\neg[x_nQx_1]$ and Q is acyclic. $\square$

Since Q is acyclic, its transitive closure is a strict partial order which (as a consequence of Szpilrajn's Theorem [9]) can be strengthened to a linear order P.[9] Let the primary criterion $f : X \to \Re$ be any representation of P. Furthermore, define the threshold map $\theta : \mathcal{D} \to \Re$ by assigning each $\theta(A) = \min_{x\in C(A)} f(x)$.

Fix a menu A and note that by construction we have $C(A) \subseteq \Gamma(A|f,\theta)$. Observe also that $x, y \in C(A)$ implies xEy and thus $g(x) = g(y)$. Hence there exists a $\overline{g} \in \Re$ such that $\forall x \in C(A)$ we have $g(x) = \overline{g}$. To establish that $\langle f, \theta, g\rangle$ is a TST representation of C, it then suffices to show that $\forall y \in \Gamma(A|f,\theta) \setminus C(A)$ we have $g(y) < \overline{g}$.

Now fix $y \in \Gamma(A|f,\theta) \setminus C(A)$, and take any $x \in C(A)$ such that $f(x) = \theta(A)$. If $g(y) > \overline{g}$, then since $g(x) = \overline{g}$ we have xQy. Alternatively, if $g(y) = \overline{g}$ then $K(x) = K(y)$ and so xEy. Since also xSy, we then have xFy and once again xQy. But from xQy it follows that xPy and hence $f(y) < f(x) = \theta(A)$, contradicting $y \in \Gamma(A|f,\theta)$. We conclude that $g(y) < \overline{g}$, and thus $\langle f, \theta, g\rangle$ is a TST representation of C.

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