# Managing pessimistic expectations and fiscal policy

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Technical Appendix

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# A Optimal wedge comparative statics

The derivations of the first-order conditions for the two- and three-period economy are subsumed in the infinite horizon economy and will not be repeated here. Consider the comparative statics that we perform by using the optimal wedge (10) and the resource constraint (1) which are repeated here for convenience,

$$(v'(1-h) - u'(c))(1+m^*\tilde{\xi} + \Phi m^*) = \Phi m^* [u''(c)c + v''(1-h)h] c+q = h.$$

Given g and  $\Phi$ , this system of equations is defining implicitly consumption and labor as functions of  $m^*$  and  $\tilde{\xi}$ ,  $c = c(m^*, \tilde{\xi})$  and  $h = h(m^*, \tilde{\xi})$ . We will sign the partial derivatives of these functions. Note at first that the resource constraint is immediately implying that  $c_i = h_i$ ,  $i = \tilde{\xi}$ ,  $m^*$ , where the subscript denotes the partial derivative. Differentiating implicitly the optimal wedge equation with respect to  $m^*$  delivers

$$c_{m^*} = h_{m^*} = \frac{\left(v'(1-h) - u'(c)\right)(\tilde{\xi} + \Phi) - \Phi\left[u''(c)c + v''(1-h)h\right]}{K},$$

where

$$K \equiv \left(u''(c) + v''(1-h)\right)(1 + m^*\tilde{\xi} + 2\Phi m^*) + \Phi m^* \left[u'''(c)c - v'''(1-h)h\right].$$
(A.1)

The numerator of  $c_{m^*}$  can be further simplified by using the optimal wedge equation to finally get,

$$c_{m^*} = h_{m^*} = \frac{\left(u'(c) - v'(1-h)\right)/m^*}{K}.$$
(A.2)

Similarly, implicitly differentiating with respect to  $\tilde{\xi}$  delivers

$$c_{\tilde{\xi}} = h_{\tilde{\xi}} = \frac{m^* (v'(1-h) - u'(c))}{K}.$$
 (A.3)

As we showed in the text, u' > v' (which implies a positive tax rate). We will work under the assumption that K < 0. Then,  $c_{m^*} = h_{m^*} < 0$  and  $c_{\tilde{\xi}} = h_{\tilde{\xi}} > 0$ , as claimed in the text. Furthermore, we can express the tax rate as a function of  $(m^*, \tilde{\xi}), \tau(m^*, \tilde{\xi}) = 1 - v'(1 - h(m^*, \tilde{\xi}))/u'(c(m^*, \tilde{\xi}))$ . Differentiating with respect to  $m^*$  and  $\tilde{\xi}$  delivers

$$\tau_i = \frac{u''(c)v'(1-h) + v''(1-h)u'(c)}{(u'(c))^2}c_i, \quad i = m^*, \tilde{\xi}.$$

Thus, since  $c_{m^*} < 0$  and  $c_{\tilde{\xi}} > 0$ , we have  $\tau_{m^*} > 0$  and  $\tau_{\tilde{\xi}} < 0$ .

**Sign of** K. We worked under the assumption that K < 0. It is convenient to decompose K as

$$K = K_c + K_h$$

where

$$K_c \equiv u''(c)(1+m^*\tilde{\xi}+2\Phi m^*)+\Phi m^* u'''(c)c$$
  

$$K_h \equiv v''(1-h)(1+m^*\tilde{\xi}+2\Phi m^*)-\Phi m^* v'''(1-h)h$$

We will show that K < 0 for a power utility function of consumption,  $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$ , and either convex marginal utility of leisure (v''' > 0) or constant Frisch elasticity,  $v(1-h) = -a_h \frac{h^{1+\phi_h}}{1+\phi_h}$ . Consider first  $K_c$ , which becomes

$$K_{c} = -\rho c^{-\rho-1} \big( 1 + m^{*} \tilde{\xi} + \Phi m^{*} (1-\rho) \big).$$

Note though that for this utility function, the first-order condition of the policy problem with respect to consumption takes the form

$$1 + m^* \tilde{\xi} + \Phi m^* (1 - \rho) = \lambda c^{\rho} > 0$$

Therefore,  $K_c < 0$ . Furthermore, if v'' > 0, then  $K_h < 0$ , since  $1 + m^* \tilde{\xi} + \Phi m^* > 0$ , as shown in footnote 12. Thus,  $K = K_c + K_h < 0$ .

Consider now the case of constant Frisch elasticity, for which the third derivative is not positive, unless  $\phi_h > 1$ , since  $v'''(1-h) = a_h \phi_h (\phi_h - 1) h^{\phi_h - 2}$ . However,  $K_h$  becomes

$$K_h = -a_h \phi_h h^{\phi_h - 1} \left[ 1 + m^* \tilde{\xi} + \Phi m^* (1 + \phi_h) \right] < 0,$$

which again delivers the desired sign of K.

**Non-separable case.** In the infinite horizon economy we treat also the non-separable case. Obviously, our comparative statics results for the separable case hold also there, by consid-

ering the derivative of consumption (labor) with respect to  $M^*$  and  $\tilde{\xi}$  (which captures now the cumulative innovation in debt). Implicitly differentiating the optimal wedge equation for non-separable utility functions (42) and the resource constraint with respect to  $(M^*, \tilde{\xi})$ delivers

$$c_{M^*} = h_{M^*} = \frac{(U_c - U_l)/M^*}{K_{\text{non}}}$$
(A.4)

$$c_{\tilde{\xi}} = h_{\tilde{\xi}} = \frac{M^*(U_l - U_c)}{K_{\text{non}}},\tag{A.5}$$

where  $K_{\rm non}$  the corresponding expression for the *non-separable* case,

$$K_{\text{non}} \equiv \left( U_{cc} - 2U_{cl} + U_{ll} \right) \left( 1 + M^* \tilde{\xi} + 2\Phi M^* \right) + \Phi M^* \left[ U_{ccc} c - U_{ccl} (2c+h) + U_{cll} (c+2h) - U_{lll} h \right].$$
(A.6)

Again, we will assume that our utility functions are such that  $K_{\text{non}} < 0$ . If there is a positive tax rate (a sufficient condition for that would be  $U_{cl} \ge 0$ ), then  $U_c > U_l$  and therefore  $c_{M^*} = h_{M^*} < 0$  and  $c_{\tilde{\xi}} = h_{\tilde{\xi}} > 0$ . The tax rate derivatives in the non-separable case are

$$\tau_i = \frac{U_{cc}U_l + U_{ll}U_c - U_{cl}(U_c + U_l)}{U_c^2}c_i, \quad i = M^*, \tilde{\xi}.$$
(A.7)

Under  $U_{cl} \ge 0$  we have  $c_{M^*} < 0$  and  $c_{\tilde{\xi}} > 0$  and the term that multiplies the consumption derivatives  $c_i$  in (A.7) is negative. Therefore,  $\tau_{M^*} > 0$  and  $\tau_{\tilde{\xi}} < 0$ .

What needs further discussion in the non-separable case is the negative sign of  $K_{\text{non}}$ . Note that when we turn off the doubts of the household by setting  $\sigma = 0$  we get  $(M^*, \tilde{\xi}) = (1, 0)$ . Thus,  $K_{\text{non}}$  at (1, 0) becomes  $K_{\text{non}}(1, 0) = (U_{cc} - 2U_{cl} + U_{ll})(1 + 2\Phi) + \Phi[U_{ccc}c - U_{ccl}(2c+h) + U_{cll}(c+2h) - U_{lll}h]$ . This would be just the second derivative of the Lagrangian of the Lucas and Stokey (1983) problem for the proper value of  $\Phi$ . In that case,  $K_{\text{non}}(1, 0) < 0$  imposes local concavity of the Lagrangian, satisfying therefore the sufficient second-order conditions of the policy problem with full confidence in the model. Therefore, for *small* doubts about the model and a  $\Phi$  close enough to the cost of distortionary taxation of Lucas and Stokey, we could justify  $K_{\text{non}} < 0$  as a sufficient condition for the satisfaction of the second-order conditions of the full confidence problem. Obviously, the same argument can be made for the separable case. Note though that for the utility functions that we used before (power in consumption and convex marginal utility of leisure or constant Frisch), we showed that K < 0 for any doubts about the model.

# B Household's inner problem and optimality conditions of the fiscal authority's problem

#### B.1 Inner problem in 3.4

Assign multipliers  $\beta^{t+1}\pi_{t+1}(g^{t+1})\rho_{t+1}(g^{t+1})$  and  $\beta^t\pi_t(g^t)\nu_t(g^t)$  on constraints (22) and (23) respectively and remember that  $M_0 \equiv 1$  and  $\pi_0(g_0) = 1$ . Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) \{ M_t(g^t) [U_t(g^t) + \theta \beta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) \ln m_{t+1}(g^{t+1})] \\ - \sum_{g_{t+1}} \beta \pi_{t+1}(g_{t+1}|g^t) \rho_{t+1}(g^{t+1}) [M_{t+1}(g^{t+1}) - m_{t+1}(g^{t+1}) M_t(g^t)] \\ - \nu_t(g^t) [\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) - 1] \}.$$

First-order necessary conditions for an interior solution are

$$m_{t+1}(g^{t+1}), t \ge 0: \qquad \nu_t(g^t) = \beta \theta M_t(g^t) [1 + \ln m_{t+1}(g^{t+1})] + \beta \rho_{t+1}(g^{t+1}) M_t(g^t) \quad (B.1)$$
  

$$M_t(g^t), t \ge 1: \qquad \rho_t(g^t) = U_t(g^t) + \beta \Big[ \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) \rho_{t+1}(g^{t+1}) + \theta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) \ln m_{t+1}(g^{t+1}) \Big]. \qquad (B.2)$$

The above conditions can be simplified as follows. Rearrange (B.1) to get

$$\ln m_{t+1}(g^{t+1}) = -\frac{\rho_{t+1}(g^{t+1})}{\theta} + \left(\frac{\nu_t(g^t)}{\beta \theta M_t(g^t)} - 1\right)$$

or

$$m_{t+1}(g^{t+1}) = \exp\left(-\frac{\rho_{t+1}(g^{t+1})}{\theta}\right) \exp\left(\frac{\nu_t(g^t)}{\beta\theta M_t(g^t)} - 1\right)$$

Taking conditional expectation of  $m_{t+1}$  and using (23) allows us to eliminate  $\nu_t(g^t)$  and get

$$m_{t+1}^*(g^{t+1}) = \frac{\exp\left(-\frac{\rho_{t+1}^*(g^{t+1})}{\theta}\right)}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp\left(-\frac{\rho_{t+1}^*(g^{t+1})}{\theta}\right)},$$
(B.3)

where the asterisks denote optimal values. Furthermore, solving forward (B.2) and imposing

the transversality condition  $\lim_{k\to\infty} \beta^k E_t M^*_{t+k} \rho^*_{t+k} = 0$  delivers

$$\rho_t^*(g^t) = \sum_{i=0}^{\infty} \sum_{g^{t+i}|g^t} \beta^i \pi_{t+i}(g^{t+i}|g^t) \frac{M_{t+i}^*(g^{t+i})}{M_t^*(g^t)} \Big[ U(g^{t+i}) + \beta\theta \sum_{g_{t+i+1}|g^{t+i}} \pi_{t+i+1}(g_{t+i+1}|g^{t+i}) m_{t+i+1}^*(g^{t+i+1}) \ln m_{t+i+1}^*(g^{t+i+1}) \Big], t \ge 1.$$

As is clear from the above condition,  $\rho_t^*(g^t)$  represents the household's utility at history  $g^t$ ,  $\rho_t^*(g^t) = V_t(g^t)$ . This fact, together with recursion (B.2) and the formula for the optimal conditional distortion (B.3), deliver the conditions in the text.

#### B.2 First-order conditions of the policy problem

The Lagrangian of the policy problem is

$$\begin{split} L &= \sum_{t=0}^{\infty} \sum_{g^{t}} \beta^{t} \pi_{t}(g^{t}) \Big\{ U(c_{t}(g^{t}), 1 - h_{t}(g^{t})) + \Phi M_{t}^{*}(g^{t}) \Omega(c_{t}(g^{t}), h_{t}(g^{t})) - \lambda_{t}(g^{t}) \big[ c_{t}(g^{t}) + g_{t} - h_{t}(g^{t}) \big] \\ &- \sum_{g_{t+1}} \beta \pi_{t+1}(g_{t+1}|g^{t}) \mu_{t+1}(g^{t+1}) \Big[ M_{t+1}^{*}(g^{t+1}) - \frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^{t}) \exp(\sigma V_{t+1}(g^{t+1}))} M_{t}^{*}(g^{t}) \big] \\ &- \xi_{t}(g^{t}) \big[ V_{t}(g^{t}) - U(c_{t}(g^{t}), 1 - h_{t}(g^{t})) - \frac{\beta}{\sigma} \ln \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^{t}) \exp(\sigma V_{t+1}(g^{t+1})) \big] \Big\} \\ &- \Phi U_{c}(c_{0}, 1 - h_{0}) b_{0}, \end{split}$$

with  $\xi_0 = 0$ ,  $M_0 = 1$  and  $g_0$  given.

Apart from first-order condition (39), the rest of the first-order conditions of the government's maximization problem can be derived in a straightforward fashion. Differentiate now the Lagrangian with respect to  $V_t(g^t)$  to get

$$V_{t}, t \geq 1: \qquad \pi_{t}(g_{t}|g^{t-1})\xi_{t}(g^{t}) = M_{t-1}^{*}(g^{t-1})\frac{\partial}{\partial V_{t}(g^{t})} \left\{ \frac{\sum_{g_{t}} \pi_{t}(g_{t}|g^{t-1}) \exp(\sigma V_{t}(g^{t})) \mu_{t}(g^{t})}{\sum_{g_{t}} \pi_{t}(g_{t}|g^{t-1}) \exp(\sigma V_{t}(g^{t}))} \right\} + \frac{\xi_{t-1}}{\sigma} \frac{\partial}{\partial V_{t}(g^{t})} \left\{ \ln \sum_{g_{t+1}} \pi_{t}(g_{t}|g^{t-1}) \exp(\sigma V_{t}(g^{t})) \right\}.$$

Note that

$$\begin{aligned} \frac{\partial}{\partial V_t(g^t)} \left\{ \frac{\sum_{g_t} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right) \mu_t(g^t)}{\sum_{g_t} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right)} \right\} &= \pi_t(g_t | g^{t-1}) \sigma \frac{\exp\left(\sigma V_t(g^t)\right)}{\sum_{g_t} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right)} \\ \cdot \left[ \mu_t(g^t) - \sum_{g_t} \pi_t(g_t | g^{t-1}) \frac{\exp\left(\sigma V_t(g^t)\right)}{\sum_{g_t} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right)} \mu_t(g^t) \right] \\ &= \pi_t(g_t | g^{t-1}) \sigma m_t^*(g^t) \left[ \mu_t(g_t) - \sum_{g_t} \pi_t(g_t | g^{t-1}) m_t^*(g^t) \mu_t(g^t) \right], \end{aligned}$$

and

$$\frac{\partial}{\partial V_t(g^t)} \left\{ \ln \sum_{g_{t+1}} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right) \right\} = \pi_t(g_t | g^{t-1}) \sigma \frac{\exp\left(\sigma V_t(g^t)\right)}{\sum_{g_t} \pi_t(g_t | g^{t-1}) \exp\left(\sigma V_t(g^t)\right)} = \pi_t(g_t | g^{t-1}) \sigma m_t^*(g^t),$$

where we used formula (24) for the household's conditional distortion. Plugging the two derivatives back to the optimality condition and simplifying delivers (39) in the text.

## C Recursive formulation

First we will give an expanded version of proposition 3 in the text.

**Proposition.** Let the approximating model of government expenditures be Markov. Then the fiscal authority's problem from period one onward can be represented recursively by keeping as a state variable the vector  $(g_t, M_t^*, \xi_t)$ . The likelihood ratio  $M_t^*$  and the multiplier  $\xi_t$  follow laws of motion

$$M_t^* = M^*(g_t, g_{t-1}, M_{t-1}^*, \xi_{t-1}; \Phi)$$
  

$$\xi_t = \xi(g_t, g_{t-1}, M_{t-1}^*, \xi_{t-1}; \Phi),$$

with initial values  $(M_0, \xi_0) = (1, 0)$ . The policy functions for consumption, household utility and debt for  $t \ge 1$  are

$$c_{t} = c(g_{t}, M_{t}^{*}, \xi_{t}; \Phi),$$
  

$$V_{t} = V(g_{t}, M_{t}^{*}, \xi_{t}; \Phi),$$
  

$$b_{t} = b(g_{t}, M_{t}^{*}, \xi_{t}; \Phi).$$

A similar recursive formulation can be achieved in terms of  $(g_t, M_t^*, \tilde{\xi}_t)$  with initial value of the state  $(g_0, 1, 0)$ .

### C.1 State variables $(M_t^*, \xi_t)$

Assume that a sequential saddle-point that solves the policy problem exists.<sup>2</sup> Our objective is to transform the sequential saddle-point into a recursive saddle-point along the lines of Marcet and Marimon (2009). To achieve that, we augment the state space and modify properly the period return function associated with the sequential saddle-point.

Fix the multiplier on the implementability constraint (32) to a positive value,  $\Phi > 0$ , and form the partial Lagrangian  $\tilde{L}_0$ 

$$\tilde{L}_0 \equiv U(g_0) + \Phi\Omega_0(g_0) - \Phi U_c(g_0)b_0 + \beta \tilde{L},$$

where

$$\tilde{L} \equiv E_0 \sum_{t=1}^{\infty} \beta^{t-1} \Big\{ U_t + \Phi M_t^* \Omega_t - \xi_t \big[ V_t - U_t - \beta (E_t m_{t+1}^* V_{t+1} + \theta E_t m_{t+1}^* \ln m_{t+1}^*) \big] \Big\}.$$

Note that we are not including in the partial Lagrangian the law of motion of the likelihood ratio  $M_t^*$  (which is the reason why we distinguish in notation between  $\tilde{L}_0$  in this section from L in section B.2) and that we have already expressed labor in terms of consumption  $h_t = c_t + g_t$  in  $\tilde{L}_0$ . Furthermore, we are differentiating between the initial period and the rest of the periods due to the presence of initial debt and the realization of uncertainty at t = 0.

Bear in mind that we have not substituted for the optimal value of the conditional likelihood ratio  $m_t^*$  (24) in the household's utility recursion, which retains *linearity* with respect to the approximating model  $\pi$  in  $\tilde{L}$ . This allows us to apply the Law of Iterated Expectations and rewrite  $\tilde{L}$  in terms of current and lagged values of  $\xi_t$ ,

$$\tilde{L} = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \Big[ U_t + \Phi M_t^* \Omega_t - \xi_t \big( V_t - U_t \big) + \xi_{t-1} \big( m_t^* V_t + \theta m_t^* \ln m_t^* \big) \Big].$$
(C.1)

Consider the saddle-point problem from period one onward,

Problem 1.

$$\min_{\xi_t,t\geq 1} \max_{c_t,m_t^*,M_t^*,V_t,t\geq 1} \tilde{L}$$

 $<sup>^{2}</sup>$ The existence of a sequential saddle-point is not guaranteed due to the non-convexity of the government's problem. However, if it exists, it solves the policy problem. See Marcet and Marimon (2009).

subject to

$$\begin{split} M_t^*(g^t) &= m_t^*(g^t) M_{t-1}^*(g^{t-1}), t \ge 1 \\ m_t^*(g^t) &= \frac{\exp(-\frac{V_t(g^t)}{\theta})}{E_{t-1}\exp(-\frac{V_t(g^t)}{\theta})}, t \ge 1, \end{split}$$

with initial values  $M_0 = 1$ ,  $\xi_0 = 0$  and  $g_0$  given.

The modified return function in (C.1) does not depend on expectations of future variables, but only on the controls  $(c_t, m_t^*, V_t, \xi_t)$  and the lagged values  $(M_{t-1}^*, \xi_{t-1})$ , which will serve as state variables. The object of interest is the value function of problem 1, which will be a solution to a saddle-point functional equation.

More precisely, assume that the approximating model of government expenditures is Markov with transition probabilities  $\pi_{g|g_-} \equiv Prob(g_t = g|g_{t-1} = g_-)$  and let the vector  $X_t \equiv (g_t, M_t^*, \xi_t)$  denote the state. Let  $W(X_-; \Phi)$  denote the corresponding value function of the saddle-point problem when the state is  $X_-$ , where the underscore "\_" stands for previous period, i.e.  $z_- \equiv z_{t-1}$  for any random variable z. The value of problem 1 is  $W(g_0, 1, 0; \Phi)$ .  $\Phi > 0$  is treated as a parameter in the value function. Then

**Bellman equation I.**  $W(\cdot; \Phi)$  satisfies the Bellman equation

$$W(g_{-}, M_{-}^{*}, \xi_{-}; \Phi) = \min_{\xi_{g}} \max_{c_{g}, m_{g}^{*}, V_{g}} \sum_{g} \pi_{g|g_{-}} \Big\{ U(c_{g}, 1 - c_{g} - g) + \Phi m_{g}^{*} M_{-}^{*} \Omega_{g} \\ -\xi_{g} (V_{g} - U(c_{g}, 1 - c_{g} - g)) + \xi_{-} (m_{g}^{*} V_{g} + \theta m_{g}^{*} \ln m_{g}^{*}) + \beta W(g, m_{g}^{*} M_{-}^{*}, \xi_{g}; \Phi) \Big\}$$

where

$$\Omega_g \equiv [U_c(c_g, 1 - c_g - g) - U_l(c_g, 1 - c_g - g)]c_g - U_l(c_g, 1 - c_g - g)g$$

and

$$m_g^* = \frac{\exp\left(-\frac{V_g}{\theta}\right)}{\sum_g \pi_{g|g_-} \exp\left(-\frac{V_g}{\theta}\right)}, \forall g$$

Time zero problem. The planner's problem at time zero takes the form

$$W_0(g_0, b_0; \Phi) = \max_{c_0} \{ U(c_0, 1 - c_0 - g_0) + \Phi \Omega_0(c_0) - \Phi U_c(c_0, 1 - c_0 - g_0) b_0 + \beta W(g_0, 1, 0; \Phi) \},\$$

which is effectively the *static* problem

$$\max_{c_0} U(c_0, 1 - c_0 - g_0) + \Phi \Omega_0(c_0) - \Phi U_c(c_0, 1 - c_0 - g_0) b_0$$

From the problem above, we get the initial period consumption,  $c_0(g_0, b_0; \Phi)$ .

Envelope conditions. The envelope conditions are

$$W_{M^*}(g_-, M^*_-, \xi_-; \Phi) = \sum_g \pi_{g|g_-} m^*_g \big[ \Phi \Omega_g + \beta W_{M^*}(g, M^*_g, \xi_g; \Phi) \big],$$
(C.2)

$$W_{\xi}(g_{-}, M_{-}^{*}, \xi_{-}; \Phi) = \sum_{g} \pi_{g|g_{-}} \left[ m_{g}^{*} V_{g} + \theta m_{g}^{*} \ln m_{g}^{*} \right].$$
(C.3)

Condition (C.3) exposes the connection between the shadow value  $\xi$  of manipulating the worst-case model and the promised utility to the household. Furthermore, solving (C.2) forward and converting to sequence notation allows us to conclude that

$$W_{M^*}(g_{t-1}, M_{t-1}^*, \xi_{t-1}; \Phi) = \Phi E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_{t-1}^*} \Omega_{t+i}$$
$$= \Phi E_{t-1} m_t^* [E_t \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_t^*} \Omega_{t+i}]$$
$$= \Phi E_{t-1} m_t^* U_{ct} b_t, \qquad (C.4)$$

where in the last line we recognized the relationship between the present value of government surpluses and debt.

**First-order conditions.** For completeness, we are going to derive the first-order conditions of the functional equation, in order to verify that they match with the first-order conditions of the sequential Lagrangian formulation. Assign the multiplier  $\pi_{g|g_-}\tilde{\mu}_g$  on the optimal distortion  $m_g^*$  and get the following first-order conditions

$$c_g: \quad (U_{l,g} - U_{c,g}) \left( 1 + \xi_g + \Phi m_g^* M_-^* \right) = \Phi m_g^* M_-^* \left[ (U_{cc} - 2U_{cl,g} + U_{ll,g}) c_g + (U_{ll,g} - U_{cl,g}) g \right]$$
(C.5)

$$m_g^*: \quad \tilde{\mu}_g = \Phi M_-^* \left[ \Omega_g + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi) \right] + \xi_- \left[ V_g + \theta (1 + \ln m_g^*) \right]$$
(C.6)

$$V_g: \quad \xi_g = \sigma m_g^* [\tilde{\mu}_g - \sum_g \pi_{g|g_-} m_g^* \tilde{\mu}_g] + m_g^* \xi_-$$
(C.7)

$$\xi_g: \quad V_g = U_g + \beta W_{\xi}(g, M_g^*, \xi_g; \Phi).$$
(C.8)

Equation (C.5) represents the familiar optimal wedge, with  $h_g = c_g + g$ . Furthermore,

using the envelope condition with respect to  $\xi$  (C.3) in optimality condition (C.8) delivers the household's utility recursion (35). It remains to show that (C.7) describes the appropriate law of motion of the multiplier  $\xi_t$ . For that consider at first (C.6) in sequence notation and use the fact that  $\ln m_t^* = -\frac{V_t}{\theta} - \ln E_{t-1} \exp\left(-\frac{V_t}{\theta}\right)$  to get

$$\tilde{\mu}_{t} = M_{t-1}^{*} \left[ \Phi \Omega_{t} + \beta W_{M^{*}}(g, M_{t}^{*}, \xi_{t}; \Phi) \right] + \xi_{t-1} \theta \left[ 1 - \ln E_{t-1} \exp(-\frac{V_{t}}{\theta}) \right].$$

Using (C.4), we see that  $\Phi \Omega_t + \beta W_{M^*}(g_t, M_t^*, \xi_t; \Phi) = \Phi(\Omega_t + \beta E_t m_{t+1}^* U_{c,t+1} b_{t+1}) = \Phi U_{ct} b_t$ . Thus

$$\tilde{\mu}_t = M_{t-1}^* \Phi U_{c_t} b_t + \xi_{t-1} \theta \left[ 1 - \ln E_{t-1} \exp\left(-\frac{V_t}{\theta}\right) \right]$$

with innovation

$$\tilde{\mu}_t - E_{t-1}m_t^*\tilde{\mu}_t = M_{t-1}^*\Phi(U_{ct}b_t - E_{t-1}m_t^*U_{ct}b_t),$$

since the term multiplying  $\xi_{t-1}$  is known with respect to information at t-1. Plugging the innovation of  $\tilde{\mu}$  in in (C.7) delivers the law of motion (44).

**Policy functions and debt.** Given the recursive representation of the government's problem, we attain a time invariant representation of the policy functions as functions of the state, e.g. the optimal policy function for consumption is  $c_g = c_g(g_-, M_-^*, \xi_-; \Phi)$ . In the case of an *i.i.d.* approximating model, we could drop the dependence on  $g_-$ . Note though that (C.5) shows that  $(g, M_g^*, \xi_g)$  is sufficient to determine c. Thus, the vector of state variables  $(g_-, M_-^*, \xi_-)$  is affecting the optimal policy for consumption at g by determining the value of the current state  $(g, M_g^*, \xi_g)$  and consequently  $c_g = c_g(g_-, M_-^*, \xi_-; \Phi) = c(g, M_g^*, \xi_g; \Phi)$ . Therefore labor and the optimal tax rate will also depend on the current values of the state. Note also that (C.8) allows us to use the same logic with the household's utility, so  $V_g = V(g, M_g^*, \xi_g; \Phi)$ . Turning to debt, using (C.4) allows us to determine the optimal debt position as a function of the current state  $b_t = b(g_t, M_t^*, \xi_t; \Phi)$ , since

$$b_t = \frac{\Omega_t}{U_{ct}} + \frac{\beta}{\Phi U_{ct}} W_{M^*}(g_t, M_t^*, \xi_t; \Phi).$$

To conclude, remember that the recursive formulation has been contingent on the value  $\Phi > 0$ . After the initial period problem and the functional problem are solved,  $\Phi$  has to be adjusted so that the intertemporal budget constraint is satisfied. The expression that we derived for optimal debt suggests the use of the derivative  $W_{M^*}$  for that purpose: Increase (decrease)  $\Phi$  if  $\frac{\Omega_0}{U_{c0}} + \frac{\beta}{\Phi U_{c0}} W_{M^*}(g_0, 1, 0; \Phi) - b_0 < (>)0$ . This procedure has to be repeated and the initial period problem and the functional equation have to be resolved till the

intertemporal budget constraint holds with equality.

# C.2 Normalized multiplier $\tilde{\xi}_t$

The same methodology allows us to derive a recursive representation in terms of the normalized multiplier  $\tilde{\xi}_t$ . Form the partial Lagrangian by multiplying the household's utility recursion (35) with  $M_t^*$  and assign to this constraint the multiplier  $\beta^t \pi_t \tilde{\xi}_t$ , with  $\tilde{\xi}_0 \equiv 0$ . Follow now similar steps as in the previous subsection to get the functional equation:

#### Bellman equation II.

$$J(g_{-}, M_{-}^{*}, \tilde{\xi}_{-}; \Phi) = \min_{\tilde{\xi}_{g}} \max_{c_{g}, m_{g}^{*}, V_{g}} \sum_{g} \pi_{g|g_{-}} \Big[ U(c_{g}, 1 - c_{g} - g) + \Phi m_{g}^{*} M_{-}^{*} \Omega_{g} \\ -m_{g}^{*} M_{-}^{*} \tilde{\xi}_{g} (V_{g} - U(c_{g}, 1 - c_{g} - g)) + \tilde{\xi}_{-} M_{-}^{*} (m_{g}^{*} V_{g} + \theta m_{g}^{*} \ln m_{g}^{*}) + \beta J(g, m_{g}^{*} M_{-}^{*}, \tilde{\xi}_{g}; \Phi) \Big],$$

where  $\Omega_g$  and  $m_g^*$  as before.

#### Envelope conditions.

$$J_{M^{*}}(g_{-}, M_{-}^{*}, \tilde{\xi}_{-}; \Phi) = \sum_{g} \pi_{g|g_{-}} \Big[ \Phi m_{g}^{*} \Omega_{g} - m_{g}^{*} \tilde{\xi}_{g}(V_{g} - U_{g}) + \tilde{\xi}_{-}(m_{g}^{*} V_{g} + \theta m_{g}^{*} \ln m_{g}^{*}) \\ + \beta m_{g}^{*} J_{M^{*}}(g, M_{g}^{*}, \tilde{\xi}_{g}; \Phi) \Big],$$
(C.9)

$$J_{\tilde{\xi}}(g_{-}, M_{-}^{*}, \tilde{\xi}_{-}; \Phi) = M_{-}^{*} \sum_{g} \pi_{g|g_{-}}(m_{g}^{*}V_{g} + \theta m_{g}^{*}\ln m_{g}^{*})$$
(C.10)

Matching first-order conditions. Assign multiplier  $\pi_{g|g_-}\hat{\mu}_g$  on the conditional distortion of the household  $m_g^*$  and derive the first-order conditions:

$$c_{g}: \quad (U_{l,g} - U_{c,g})(1/M_{g}^{*} + \tilde{\xi}_{g} + \Phi) = \Phi \left[ (U_{cc,g} - 2U_{cl,g} + U_{ll,g})c_{g} + (U_{ll,g} - U_{cl,g})g \right]$$
(C.11)

$$m_g^*: \quad \hat{\mu}_g = M_-^* \Big[ \Phi \Omega_g - \tilde{\xi}_g (V_g - U_g) + \tilde{\xi}_- (V_g + \theta (\ln m_g^* + 1)) \Big]$$

$$+\beta J_M^*(g, M_g^*, \tilde{\xi}_g; \Phi) \Big]$$
(C.12)

$$V_g: \quad \tilde{\xi}_g M_-^* = \sigma(\hat{\mu}_g - \sum_g \pi_{g|g_-} m_g^* \hat{\mu}_g) + \tilde{\xi}_- M_-^*$$
(C.13)

$$\tilde{\xi}_g: \quad m_g^* M_-^* V_g = m_g^* M_-^* U_g + \beta J_{\tilde{\xi}}(g, M_g^*, \tilde{\xi}_g; \Phi)$$
(C.14)

Condition (C.11) describes the familiar optimal wedge. Turn now into sequence notation, update the envelope condition (C.10) one period, substitute in (C.14) and simplify to get

the household's utility recursion,

$$V_t = U_t + \beta (E_t m_{t+1}^* V_{t+1} + \theta E_t m_{t+1}^* \ln m_{t+1}^*).$$

There is some work needed in order to derive the law of motion of the multiplier  $\tilde{\xi}_t$  in the text. Consider the envelope condition (C.9) and solve it forward to get

$$J_{M^*}(g_{t-1}, M_{t-1}^*, \tilde{\xi}_{t-1}; \Phi) = \Phi E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_{t-1}^*} \Omega_{t+i}$$
$$-E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_{t-1}^*} \tilde{\xi}_{t+i} (V_{t+i} - U_{t+i})$$
$$+E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i-1}^*}{M_{t-1}^*} \tilde{\xi}_{t+i-1} (m_{t+i}^* V_{t+i} + \theta m_{t+i}^* \ln m_{t+i}^*)$$

The last sum in the third line can be rewritten as

$$E_{t-1} \sum_{i=0}^{\infty} \beta^{i} \frac{M_{t+i-1}^{*}}{M_{t-1}^{*}} \tilde{\xi}_{t+i-1} (m_{t+i}^{*} V_{t+i} + \theta m_{t+i}^{*} \ln m_{t+i}^{*}) = \tilde{\xi}_{t-1} E_{t-1} (m_{t}^{*} V_{t} + \theta m_{t}^{*} \ln m_{t}^{*})$$
$$+ E_{t-1} \sum_{i=0}^{\infty} \beta^{i} \tilde{\xi}_{t+i} \beta (m_{t+i+1}^{*} V_{t+i+1} + \theta m_{t+i+1}^{*} \ln m_{t+i+1}^{*}).$$

Thus the derivative of the value function with respect to the likelihood ratio  $M^*$  becomes

$$\begin{aligned} J_{M^*}(g_{t-1}, M_{t-1}^*, \tilde{\xi}_{t-1}; \Phi) &= \Phi E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_{t-1}^*} \Omega_{t+i} + \tilde{\xi}_{t-1} E_{t-1}(m_t^* V_t + \theta m_t^* \ln m_t^*) \\ &- E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M_{t+i}^*}{M_{t-1}^*} \tilde{\xi}_{t+i} \Big( V_{t+i} - U_{t+i} - \beta E_{t+i} \Big( m_{t+i+1}^* V_{t+i+1} + \theta m_{t+i+1}^* \ln m_{t+i+1}^* \Big) \Big) \\ &= \Phi E_{t-1} m_t^* U_{ct} b_t + \tilde{\xi}_{t-1} E_{t-1}(m_t^* V_t + \theta m_t^* \ln m_t^*), \end{aligned}$$

by using the household's utility recursion and the relationship between debt and the present value of future government surpluses.

Update  $J_{M^*}$  one period and plug it in the first-order condition (C.12) to get

$$\begin{aligned} \hat{\mu}_{t} &= M_{t-1}^{*} \Big[ \Phi \big( \Omega_{t} + \beta E_{t} m_{t+1}^{*} U_{c,t+1} b_{t+1} \big) \\ &- \tilde{\xi}_{t} \big( V_{t} - U_{t} - \beta (E_{t} m_{t+1}^{*} V_{t+1} + \theta E_{t} m_{t+1}^{*} \ln m_{t+1}^{*}) \big) \\ &+ \tilde{\xi}_{t-1} \big( V_{t} + \theta (\ln m_{t}^{*} + 1) \big) \Big] \\ &= M_{t-1}^{*} \Big[ \Phi \big( \Omega_{t} + \beta E_{t} m_{t+1}^{*} U_{c,t+1} b_{t+1} \big) + \tilde{\xi}_{t-1} \big( V_{t} + \theta (\ln m_{t}^{*} + 1) \big) \Big], \end{aligned}$$

using again the household's utility recursion. Note that  $\Omega_t + \beta E_t m_{t+1}^* U_{c,t+1} b_{t+1} = U_{ct} b_t$ . Use now the expression for the conditional distortion  $m_t^*$  to finally get

$$\hat{\mu}_t = M_{t-1}^* \Big[ \Phi U_{ct} b_t + \tilde{\xi}_{t-1} \theta (1 - \ln E_{t-1} \exp(\sigma V_t)) \Big].$$

Therefore, the innovation in  $\hat{\mu}_t$  becomes  $\hat{\mu}_t - E_{t-1}m_t^*\hat{\mu}_t = \Phi M_{t-1}^*[U_{ct}b_t - E_{t-1}m_t^*U_{ct}b_t]$ . Plugging the innovation in (C.13) and simplifying delivers the law of motion of the normalized multiplier (46).