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Managing pessimistic expectations and fiscal policy

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This paper studies the design of optimal fiscal policy when a government that fully trusts the probability model of government expenditures faces a fearful public that forms pessimistic expectations. We identify two forces that shape our results. On the one hand, the government has an incentive to concentrate tax distortions on events that it considers unlikely relative to the pessimistic public. On the other hand, the endogeneity of the public's expectations gives rise to a novel motive for expectation management that aims toward the manipulation of equilibrium prices of government debt in a favorable way. These motives typically act in opposite directions and induce persistence to the optimal allocation and the tax rate.

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1. Introduction

Optimal policy design problems routinely exploit the rational expectations assumption that attributes a unique and fully trusted probability model to all agents. That useful assumption precludes carrying out a coherent analysis that attributes fears of model misspecification to some or all agents.

This paper studies the design of optimal fiscal policy in an environment where the public has doubts about the probability model of exogenous government expenditures and guards itself against this ambiguity by forming pessimistic expectations. In contrast,

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the fiscal authority or government (these two terms are used interchangeably throughout the paper) completely trusts the probability model and uses it to assess the likelihood of shocks in the design of fiscal policy.

We are motivated by situations where fearful markets constrain the actions of fiscal authorities, as in the recent European fiscal crisis, for example. In various cases, fiscal authorities take actions to alleviate market pressures and try to convince markets that fiscal policies are sustainable. Attempts to manage fearful expectations in such environments raise natural questions about how this is possible and about how fiscal policy should be designed. This paper consists of a first take on a theoretical model that explores this type of questions by positing a government that shows full confidence in the probability model of government expenditures, whereas the public does not.[1]

The distinctive feature of our approach is the fact that the agents' fears of model misspecification cause them to twist their expectations about exogenous shocks in an *endogenous* way by assigning high probability to low utility events and low probability to high utility events. The fiscal authority, by its choice of tax and debt policies, affects the agents' utility and, therefore, their cautious beliefs. As a result, this paper features a notion of *expectation management* that is absent from the standard rational expectations paradigm, where beliefs about exogenous shocks are fixed and actually correct.[2]

For our analysis, we adopt the complete-markets economy without capital that is analyzed by Lucas and Stokey (1983), but we modify the representative household's preferences to express its concerns about misspecification of the stochastic process for government expenditures. The Lucas and Stokey (1983) setup consists of the canonical framework for analyzing optimal fiscal policy when lump-sum taxes are not available.[3] There is an exogenous stream of government expenditures that the government has to finance in the least distortionary way through a linear tax on labor income or (and) by issuing state-contingent debt. Our household expresses model distrust by ranking consumption and leisure plans according to the multiplier preferences of Hansen and Sargent (2001); only when a multiplier parameter assumes a special value do the expected utility preferences of Lucas and Stokey (1983) emerge as a special case in which the representative household completely trusts its probability model.[4] The government shows complete confidence in the probability model of government expenditures and acts *paternalistically* by using it in the evaluation of the household's expected utility.[5] The endogenous household's beliefs play a crucial role in our analysis, because they affect the

[1]Lack of confidence in models also seems to have become pronounced in the recent financial crisis. See, for example, Caballero and Krishnamurthy (2008), Caballero and Kurlat (2009), and Uhlig (2010).

[2]The management of pessimistic expectations could also have alternative interpretations in terms of risk-sensitive preferences.

[3]The European fiscal crisis serves as a motivating example of an environment with fearful expectations. We do not try to capture default here.

[4]Multiplier preferences lead to tractable functional forms. See Maccheroni et al. (2006a, 2006b) and Strzalecki (2011) for axioms that rationalize multiplier preferences as expressions of model ambiguity.

[5]Karantounias (2011) studies alternative sets of assumptions that allow the government to doubt the probability model either more or less than the household and also possibly instructs the government to evaluate expected utilities using the representative household's beliefs, becoming in that special case the *Ramsey* planner. The current setup isolates key forces that also operate in that alternative setting.

equilibrium price of government debt and, therefore, the need to resort to distortionary taxation.

There are two main forces that operate in our setup. The first reflects the paternalistic motives that the fiscal authority exhibits by bestowing full confidence in the probability model of government expenditures. At a casual level, one would think that a paternalistic fiscal authority with no doubts about the model would like the household to hold the same expectations. The proper way to think about our setup is in terms of the optimal allocation of tax distortions. We find that the fiscal authority has an incentive to tax more contingencies that it considers less probable than the household and less contingencies that it considers more probable than the household. The reason behind this is straightforward. In the eyes of the fiscal authority, the welfare loss from taxing contingencies that it considers unlikely relative to the household is *small*, motivating it to shift taxes toward these contingencies. Thinking in terms of asset prices, claims on contingencies that the household considers likely (and the government does not) command an inflated price in the eyes of the government. This *mispricing* prompts the government to sell more debt (or buy less assets) at these expensive prices and, therefore, to tax these particular contingencies more.[6] The low utility events to which the household assigns high probability are typically associated with high government expenditures. Thus, the paternalism of the fiscal authority is expressed as an incentive to tax more (less) when government expenditures are high (low).

However, as we stressed earlier, the defining characteristic of our analysis of model uncertainty is the endogeneity of the worst-case beliefs of the household and their effect on asset prices, a feature that creates a separate and distinct mechanism from the paternalistic motives analyzed previously. The government, by choosing a low tax rate at a particular contingency, increases the utility of the household and, therefore, it decreases the household's assessment of the likelihood of this contingency. As a result, the price of a contingent claim decreases. The government has an incentive to use this mechanism to make claims *cheaper* when it purchases ex ante assets. In contrast, the government increases the tax rate so as to increase the price of a contingent claim through this mechanism when it is *selling* claims (issuing debt) to reduce the return on debt. The government is typically hedging shocks by purchasing ex ante assets contingent on high government expenditures to finance deficits and is selling debt contingent on low government expenditures that is paid for by running a surplus. Therefore, the government's *price manipulation* through the household's expectations leads to an incentive to tax less (more) when government expenditures are high (low), contributing an opposite and offsetting force to the paternalistic force.

The two forces that we describe also affect the dynamics of the optimal plan by introducing dependence on the past *history* of shocks, whereas the full confidence economy of Lucas and Stokey (1983) prescribes history independence.[7] Consider first the

[6]We are thankful to the co-editor and an anonymous referee for stressing the mispricing interpretation.

[7]It is interesting to observe that although history dependence is not our aim in this paper, it emerges due to the paternalistic and the price manipulation motives of the fiscal authority, despite the complete markets assumption. Aiyagari et al. (2002) obtain history dependence in a Ramsey problem, and Battaglini and Coate (2008) in a political–economic bargaining equilibrium by dropping complete markets.

paternalistic force. The paternalistic-mispricing motive depends on the household's probability assessment of the entire history of shocks up to the current period and not solely on the probability of the current shock. Assume, for example, that there was a high shock in the past, an event to which the cautious household assigns high probability, therefore, motivating the fiscal authority to tax high in the past. However, the probability of the history of shocks that includes this shock in the past and all shocks up to the current period increases as well, creating an incentive to also tax high in the current period. As a result, there is an incentive to *keep* the tax rate high (low) *following* a high (low) shock, inducing persistence to the optimal tax rate.

In a sense, the history dependence arising from the paternalistic motives of the government is due to the backward-looking nature of the discrepancy between the government's and the household's beliefs. Turning to the price manipulation efforts of the government, we find that they also introduce history dependence, but it is due to the *forward-looking* nature of the household's endogenous beliefs. The household is forward-looking when it forms its worst-case beliefs by taking into account *both* period utility *and* the discounted value of future utility. As a result, a low tax rate in the future, by increasing future utility, also increases current utility. This forces the fiscal authority to take into account the past when it chooses the future tax rate. In particular, if there was an incentive to set a low tax rate in the past (as in the case of a high shock), this incentive will persist in the future. Therefore, the price manipulation motives of the fiscal authority make it *keep* the tax rate low (high) *following* a high (low) shock. The marginal incentives of managing the household's pessimistic expectations are tracked by the entire history of government debt or asset positions, since they identify the incentives of increasing or decreasing asset prices, respectively, along the history of shocks.

To conclude, another take on the tension between the government's two opposite incentives can be illustrated by considering a sequence of high shocks that are associated with low utility every period. The pessimistic household assigns an increasingly higher probability to the partial history of shocks over time, therefore, leading to an increasing sequence of tax rates due to the paternalistic motive of the government; in other words, to a *back-loading* of taxes. If the government hedges these high shocks by buying assets each period, the price manipulation motive leads though to a decreasing sequence of taxes over time; in other words, a *front-loading* of taxes. Note that without doubts about the model, the tax rate would stay constant over time in the face of a sequence of high shocks.

1.1 *Related literature*

The policy problem that we formulate is a Stackelberg problem with a leader who trusts the model and a follower who has doubts about it. Analysis of such problems is novel to our knowledge and comprises a methodological contribution of this paper. Robust control in forward-looking models is analyzed by Hansen and Sargent (2008, Chap. 16), who formulate a model in which a Stackelberg leader distrusts an approximating model, while a competitive fringe of followers completely trusts it. The reverse assumptions about specification concerns that we make here alter the policy problem nontrivially by necessitating consideration of the follower's utility recursion, to enable the follower's endogenous worst-case beliefs to be determined. We tackle this problem by applying recursive methods along the lines of Marcet and Marimon (2011). These methods could potentially be applied in different policy settings where followers have doubts about the model.

Other contributions that share our aim of attributing misspecification fears to at least some agents include Kocherlakota and Phelan (2009), who study a mechanism design problem using a max-min expected utility criterion, and Barlevy (2009, 2011), who studies policy makers with fears of model misspecification.[8] Woodford (2010), the most interesting previous paper in several ways, sets up a particular timing to conceal the private sector's beliefs from the government. In Woodford's model, both the government and the private sector fully trust their *own* models, but the government distrusts its knowledge of the private sector's beliefs about prices. Arranging things so that this is possible is subtle because, with enough markets, equilibrium prices and allocations reveal private sector beliefs. In contrast to Woodford, we set things up with complete markets whose prices fully reveal private sector beliefs to the fiscal authority.

Any analysis with multiple subjective probability models requires a convenient way to express those models. Along with Woodford (2010), this paper uses the martingale representation of Hansen and Sargent (2005, 2007) and Hansen et al. (2006). From the point of view of the approximating model, these martingale perturbations look like multiplicative preference shocks. In the present context, the fiscal authority manipulates those "shocks."

This paper resides at the intersection of three literatures. Optimal policy analysis by Bassetto (1999), Chari et al. (1994), Zhu (1992), Angeletos (2002), and Buera and Nicolini (2004) in complete markets or, in incomplete markets, by Aiyagari et al. (2002), Shin (2006), and Marcet and Scott (2009), and recursive representations as in Chang (1998) and Sleet and Yeltekin (2006) are all relevant antecedents of our work. The multiplier preferences we use are closely related to risk-sensitive preferences and to Epstein and Zin (1989) and Weil (1990) preferences, and, therefore, our work is also related to Anderson (2005) and Tallarini (2000), who study the impact of risk-sensitivity on risk-sharing and on business cycles, respectively, as well as to Hansen et al. (1999), who study the effect of doubts about the model on permanent income theory and asset prices. Another related line of work is Farhi and Werning (2008), who analyze the implications of recursive preferences in private information settings.

1.2 *Organization*

Section 2 features a two-period version of our economy that illustrates the paternalistic and the price manipulation motives. In the same section, we also analyze a three-period economy, so as to clarify the dependence of the optimal plan on the past. In Section 3, we lay out the infinite horizon economy. Sections 4 and 5 show the natural generalization of our results in an infinite horizon, the novel intertemporal smoothing motives

[8]This work is also linked in a general sense to that of Brunnermeier et al. (2007), who study a setting in which households choose their beliefs.

that arise, and the recursive representation of the policy problem. Section 6 concludes. The Appendix provides information to the reader on the technical aspects of the policy problem.

2. The basic forces

The basic forces of our model are best captured in a two-period economy. Afterward, we proceed to a three-period economy, so as to illustrate the history dependence of the optimal plan.[9]

2.1 *A two-period economy*

We adopt a two-period version of the Lucas and Stokey (1983) economy without capital, with a representative household that fears model misspecification. There are two periods, $t=0$ and $t=1$. At period zero, there is no production, consumption, or initial debt. At period one there is uncertainty captured by the realization of an exogenous government expenditure shock g_1 that takes a finite number of values. Markets are complete and competitive. The household consumes $c_1(g_1)$ and has one unit of time that it allocates between work $h_1(g_1)$ and leisure $l_1(g_1)$. There is a linear technology in labor with productivity normalized to unity. The resource constraint at period one is

$$c_1(g_1)+g_1=h_1(g_1), \quad \forall g_1. \tag{1}$$

Competition makes the real wage equal to unity, $w_1(g_1)=1$, for all g_1. The household is taxed linearly on its labor income with tax rate $\tau_1(g_1)$ and trades with the government at $t=0$ claims contingent on the realization of g_1 with price $q_1(g_1)$. The household's budget constraint at $t=0$ reads

$$\sum_{g_1} q_1(g_1)c_1(g_1) \leq \sum_{g_1} q_1(g_1)(1-\tau_1(g_1))h_1(g_1), \tag{2}$$

whereas the government budget constraint reads

$$\sum_{g_1} q_1(g_1)(\tau_1(g_1)h_1(g_1)-g_1) \geq 0.$$

Model misspecification. The representative household and the government share an approximating probability model of government expenditures in the form of $\pi_1(g_1)$. The household is afraid that the probability measure is misspecified and considers alternative probability measures $\tilde{\pi}_1$ that are absolutely continuous with respect to π_1. Absolute continuity means that events that receive positive probability under the alternative model, also receive positive probability under the approximating model. We express these alternative models with a nonnegative random variable $m_1(g_1) \equiv \tilde{\pi}_1(g_1)/\pi_1(g_1) \geq 0$ that is interpreted as a likelihood ratio. The likelihood ratio m_1 integrates to unity with

[9] We are grateful to an anonymous referee for suggesting this route as the most effective way to convey the essence of our results.

respect to the approximating model, $\sum_{g_1} \pi_1(g_1)m_1(g_1) = 1$. The discrepancy between the alternative model and the approximating model is measured in terms of relative entropy,

$$\varepsilon(m) \equiv \sum_{g_1} \pi_1(g_1)m_1(g_1) \ln m_1(g_1).$$

Note that relative entropy is zero if the approximating and the alternative model coincide, and is positive otherwise. Relative entropy is the expected log-likelihood ratio under the alternative model.

The household expresses its aversion to model misspecification by using the multiplier preferences of Hansen and Sargent (2001),

$$\min_{m_1(g_1)\geq 0} \sum_{g_1} \pi_1(g_1)m_1(g_1)\big[u(c_1(g_1)) + v(1-h_1(g_1))\big] + \theta \sum_{g_1} \pi_1(g_1)m_1(g_1) \ln m_1(g_1),$$

subject to $\sum_{g_1} \pi_1(g_1)m_1(g_1) = 1$. We assume that the utility functions of consumption and leisure are strictly monotonic, strictly concave, and thrice continuously differentiable. The household's doubts about the model π are captured by the penalty parameter $\theta > 0$. Higher values of θ represent more confidence in the approximating model π. Full confidence is captured by $\theta = \infty$, which reduces the above preferences to the expected utility preferences of Lucas and Stokey (1983). We assume separability between consumption and leisure just for the two- and three-period economy. We subsequently restore nonseparabilities between consumption and leisure in the infinite horizon economy.

Household's problem. The household's problem is

$$\max_{c_1(g_1),h_1(g_1)} \min_{m_1(g_1)\geq 0} \sum_{g_1} \pi_1(g_1)m_1(g_1)\big(u(c_1(g_1)) + v(1-h_1(g_1))\big) + \theta \sum_{g_1} \pi_1(g_1)m_1(g_1) \ln m_1(g_1)$$

subject to the budget constraint (2), the nonnegativity constraint for consumption $c_1(g_1) \geq 0$, the feasibility constraint for labor $h_1(g_1) \in [0,1]$, and the constraint that m_1 has to integrate to unity.

Worst-case beliefs. Consider first the inner problem that minimizes the utility of the household subject to the restriction that m_1 integrates to unity.[10] The optimal distortion is indicated with an asterisk and takes the exponentially twisting form

$$m_1^*(g_1) = \frac{\exp(-(u(c_1(g_1)) + v(1-h_1(g_1)))/\theta)}{\sum_{g_1} \pi_1(g_1)\exp(-(u(c_1(g_1)) + v(1-h_1(g_1)))/\theta)}, \quad \forall g_1. \tag{3}$$

Equation (3) denotes that the household assigns high probability to low utility events and low probability to high utility events. By depending on utility, the household's worst-case beliefs become *endogenous*. As a result, the actions of the government, by determining the household's utility, affect its worst-case beliefs.

[10] See the Appendix for details of the derivations.

Furthermore, inserting the optimal distortion m_1^* into the preferences of the household delivers the indirect utility function

$$\sigma^{-1} \ln \sum_{g_1} \pi_1(g_1) \exp\big(\sigma\big(u(c_1(g_1)) + v(1 - h_1(g_1))\big)\big),$$

where $\sigma \equiv -1/\theta < 0$.

Proceeding to the first-order conditions of the maximization problem, we get the intratemporal labor supply condition

$$\frac{v'(1 - h_1(g_1))}{u'(c_1(g_1))} = 1 - \tau_1(g_1)$$

and the optimality condition for the allocation of consumption between state g_1 and $\hat{g}_1$,

$$\frac{q_1(\hat{g}_1)}{q_1(g_1)} = \frac{\pi_1(\hat{g}_1)}{\pi_1(g_1)} \frac{m_1^*(\hat{g}_1)}{m_1^*(g_1)} \frac{u'(c_1(\hat{g}_1))}{u'(c_1(g_1))},$$

which equates the relative price to the ratio of the worst-case beliefs times the ratio of marginal utilities. Note how the endogenous worst-case beliefs show up in the determination of asset prices. This is the *novel* channel that the fiscal authority exploits to finance the exogenous government expenditures.

The competitive equilibrium given taxes τ_1 is characterized by the household's optimality conditions and budget constraint together with the resource constraint (1).

2.1.1 *Optimal taxation* Following Lucas and Stokey (1983), we employ the primal approach, and eliminate equilibrium prices and tax rates from the household's budget constraint (2). This delivers the implementability constraint

$$\sum_{g_1} \pi_1(g_1) m_1^*(g_1) \big[u'(c_1(g_1)) c_1(g_1) - v'(1 - h_1(g_1)) h_1(g_1)\big] = 0. \tag{4}$$

In describing the economy, we use the *intertemporal* budget constraint of the government (which holds with equality at equilibrium). The period government budget constraint at $t = 0$ is

$$\sum_{g_1} q_1(g_1) b_1(g_1) = 0,$$

and at $t = 1$, when the shock takes the value g_1,

$$b_1(g_1) = \tau_1(g_1) h_1(g_1) - g_1.$$

The government surplus or deficit equals, in equilibrium, the household's consumption net of after-tax labor income. If $b_1(g_1) > 0$, then at period $t = 0$, the government issues debt for contingency g_1 that is paid back by a surplus. If $b_1(g_1) < 0$, then at $t = 0$, the government buys assets (household liabilities) that are used to finance a deficit at g_1. The term $u'(c)c - v'(1 - h)h$ in (4) expresses the government's asset position in marginal utility of consumption terms, $u'(c_1)b_1$. The implementability constraint (4) equates the present value of government surpluses to the initial debt (which is zero).

As we discussed in the Introduction, the fiscal authority has full confidence in the probability model of government expenditures and acts *paternalistically*, i.e., it imposes its own, full-confidence expected utility criterion when it ranks alternative consumption–leisure plans (c, l).

Definition 1. The fiscal authority's problem is

$$\max_{\{c_1(g_1), h_1(g_1), m_1^*(g_1)\}} \sum_{g_1} \pi_1(g_1)\big(u(c_1(g_1)) + v(1 - h_1(g_1))\big)$$

subject to (4), the resource constraint (1), and the endogenous worst-case beliefs (3) of the household.

Assign multipliers Φ on the implementability constraint (4), assign $\pi_1(g_1)\lambda_1(g_1)$ on the resource constraint (1), and assign $\pi_1(g_1)\mu_1(g_1)$ on the worst-case distortion (3). The first-order conditions for an interior solution are

$$c_1(g_1):\quad u'(c_1(g_1))(1 + \Phi m_1^*(g_1) + \sigma m_1^* \eta_1(g_1)) + \Phi m_1^*(g_1) u''(c_1(g_1)) c_1(g_1) = \lambda_1(g_1) \tag{5}$$

$$h_1(g_1):\quad -v'(1 - h_1(g_1))(1 + \Phi m_1^*(g_1) + \sigma m_1^*(g_1)\eta_1(g_1)) + \Phi m_1^*(g_1) v''(1 - h_1(g_1)) h_1(g_1) = -\lambda_1(g_1) \tag{6}$$

$$m_1^*(g_1):\quad \mu_1(g_1) = \Phi\big[u'(c_1(g_1))c_1(g_1) - v'(1 - h_1(g_1))h_1(g_1)\big], \tag{7}$$

where

$$\eta_1(g_1) \equiv \mu_1(g_1) - \sum_{g_1} \pi_1(g_1) m_1^*(g_1) \mu_1(g_1),$$

the *innovation* in μ_1 under the household's distorted measure. Obviously,

$$\sum_{g_1} \pi_1(g_1) m_1^*(g_1) \eta_1(g_1) = 0.$$

Before we proceed to an analysis of the optimal government policy, it is helpful to consider the derivation of the first-order condition with respect to consumption (5), which is rewritten as

$$u'(c_1) + \underbrace{\Phi m_1^*[u''(c_1)c_1 + u'(c_1)]}_{\text{effect on government surplus in MU terms}} + \underbrace{m_1^* \sigma u'(c_1) \eta_1}_{\text{effect on endogenous beliefs } m^*} = \underbrace{\lambda_1}_{\text{shadow value of output}}, \tag{8}$$

where we dropped the argument g_1 for notational simplicity. An increase in consumption provides to the fiscal authority marginal utility $u'(c)$ that is captured by the first term in (8). Note that the marginal utility is not multiplied with the worst-case likelihood ratio m_1^* since the fiscal authority has *full* confidence in the probabilistic model. The second term captures the effect that an increase in consumption has on the government surplus in marginal utility terms. The third term represents the effect of an increase in consumption on the *endogenous* likelihood ratio m_1^*: an increase in consumption leads to an increase in utility and, therefore, to a reduction in m_1^*, which is captured by term $\sigma u'(c) < 0$, since $\sigma < 0$. This term is multiplied by η_1, the innovation in the shadow value μ_1 of changing the likelihood ratio m_1^*, that summarizes the marginal benefits or costs of affecting the household's beliefs. The shadow value μ_1 and the innovation η_1 are analyzed in detail later. The sum of these three terms should equal the shadow value of output λ_1. Analogous interpretations hold for the first-order condition with respect to labor (6).

Optimal wedge and tax rate. Combine (5) and (6) to eliminate λ_1 and get an expression for the *optimal wedge*,

$$\begin{aligned} &v'(1-h_1(g_1)) - u'(c_1(g_1)) \\ &\quad = \frac{\Phi}{1/m_1^*(g_1) + \tilde{\xi}_1(g_1) + \Phi}\big[u''(c_1(g_1))c_1(g_1) + v''(1-h_1(g_1))h_1(g_1)\big], \end{aligned} \tag{9}$$

where we define $\tilde{\xi}_1 \equiv \sigma\eta_1$.[11] By using $\tau_1 = 1 - v'(1-h_1)/u'(c_1)$, the optimal wedge equation can be rearranged to get an expression for the tax rate (dropping g_1 again),

$$\tau_1 = \frac{\Phi}{1/m_1^* + \tilde{\xi}_1 + \Phi(1+\epsilon_{h,1})}[\gamma_{\text{RA},1} + \epsilon_{h,1}]. \tag{10}$$

Here $\gamma_{\text{RA},1}$ stands for the coefficient of relative risk aversion $\gamma_{\text{RA},1} \equiv -u''(c_1)c_1/u'(c_1)$ and $\epsilon_{h,1}$ stands for the elasticity of the marginal disutility of labor, $\epsilon_{h,1} \equiv -v''(1-h_1)h_1/v'(1-h_1)$. Given the concavity of the utility function, (10) shows that the tax rate is positive for every contingency (and, therefore, $u' > v'$), as long as there is a need for distortionary taxation, which is captured by the multiplier $\Phi > 0$.[12]

2.1.2 *The two forces* In the full confidence economy of Lucas and Stokey (1983) ($\sigma = 0$), we have $m_1^* \equiv 1$ and $\tilde{\xi}_1 \equiv 0$. Our setup gives rise to two deviations from the Lucas and Stokey framework, as the optimal wedge equation (9) shows: the ratio of the government's over the household's worst-case beliefs $1/m_1^*$, which captures the paternalistic motive of the government, and $\tilde{\xi}_1$, which captures the price manipulation through the household's endogenous worst-case beliefs.

Typically, we expect the pessimistic household to assign high probability on states where government expenditures are high and low probability on states where government expenditures are low, so we expect $m_1^* > 1$ when g_1 is high and $m_1^* < 1$ when g_1

[11] Assume that we write the worst-case beliefs of the household as $m_1^* = \exp(\sigma V_1)/\sum \pi_1 \exp(\sigma V_1)$, where $V_1 = u(c_1) + v(1-h_1)$ and that we assign the multiplier $\pi_1\xi_1$ on the additional constraint that equates V_1 to current utility of consumption and leisure. So ξ_1 captures the shadow value to the government of the household's utility. Then we have an additional first-order condition with respect to V_1 that equates the shadow value of utility to a multiple of the innovation η_1, $\xi_1 = \sigma m_1^* \eta_1$. This leads to $\tilde{\xi}_1 = \sigma\eta_1$, by defining the *normalized* multiplier $\tilde{\xi}_1 \equiv \xi_1/m_1^*$. This construction is redundant in the two-period economy but it proves useful in the three-period and in the infinite horizon economy.

[12] Positivity of the tax rate is established by showing that the denominator in (10) is positive despite the presence of η_1, which can take both positive and negative values. Use the definition of $\tilde{\xi}_1$ and rearrange (5) to get $1/m_1^* + \tilde{\xi}_1 + \Phi = m_1^{*-1}[\lambda_1 - \Phi m_1^* u''(c_1)c_1]/u'(c_1) > 0$, since $\lambda_1 > 0$.

is low. Furthermore, note from the first-order condition (7) that μ_1 is equal to a multiple of the government surplus in marginal utility terms, $\mu_1 = \Phi u'(c_1)b_1$, and, therefore, $\sum_{g_1} \pi_1(g_1)m_1^*(g_1)\mu_1(g_1) = \Phi \sum_{g_1} \pi_1(g_1)m_1^*(g_1)u'(c_1(g_1))b_1(g_1) = 0$, since the present value of government surpluses is zero. Thus, $\eta_1 = \mu_1$ and $\tilde{\xi}_1 \equiv \sigma\eta_1 = \sigma\Phi u'(c_1)b_1$. If the government hedges shocks by issuing debt contingent on low g_1 and buying assets contingent on high g_1, then we expect $\tilde{\xi}_1 < 0$ for low g_1 and $\tilde{\xi}_1 > 0$ for high g_1.

In the next paragraphs we provide a detailed interpretation of the economic forces at play.

Paternalism. Turn first to the effect of asymmetry in evaluating welfare between the government and the household. The optimal wedge equation (9) shows that an *increase* in m_1^* leads to a *decrease* in consumption and labor, and, therefore, to an *increased* tax rate, keeping everything else equal.[13] Thus, the fiscal authority has the incentive to tax more (less) contingencies that it considers relatively less (more) probable than the household. The intuition behind this result is straightforward. A high tax rate on a state of the world that the government considers unlikely relative to the household implies a small welfare loss, creating the incentive to concentrate distortions on those states. From a different perspective, the government reacts to what it sees as mispricing, by taxing more and buying—at inflated prices—fewer assets (or issuing more debt) contingent on the states of the world that it considers unlikely relative to the household. In contrast, the government taxes less and buys more assets (issue less debt) contingent on the states of the world that it considers more likely than the household. Associating the low-utility events to which the household assigns high probability with high government expenditures leads to the conclusion that paternalism creates an incentive to tax more when there is a high government expenditure shock and less when there is a low government expenditure shock.

Price manipulation through expectation management. The government, by increasing consumption at a particular state of the world, increases the household's utility and, therefore, leads the household to decrease the probability that it assigns to this state. As a result, the price of a claim contingent on this state *decreases*. The marginal benefits or costs of affecting asset prices through this mechanism are captured by the multiplier μ_1, which measures the marginal benefit of increasing m_1^*. As noted before, μ_1 is just equal to the government surplus or deficit in marginal utility terms (which equals maturing government debt or assets in marginal utility terms) times the cost of distortionary taxation Φ, $\mu_1 = \Phi u'(c_1)b_1$. Therefore, there is a marginal benefit of increasing m_1^* (and the price of the respective state-contingent claim) when the government is issuing at $t = 0$ debt for contingency g_1 ($b_1(g_1) > 0$) and a marginal cost when it buys assets that are due at contingency g_1 ($b_1(g_1) < 0$). In a first-best world ($\Phi = 0$), μ_1 is zero, capturing the fact that asset prices are irrelevant for taxation purposes in a world where lump-sum taxes are available. The intuition behind the government's price manipulation is as follows. The government has a marginal incentive to increase asset prices in situations where it

[13]See the Appendix for the conditions under which this claim holds. These conditions are satisfied for a power utility function of consumption and either convex marginal utility of leisure or a disutility function of labor with constant Frisch elasticity.

sells claims ($b_1 > 0$), so as to decrease the return on state-contingent debt. Alternatively, in situations when the government is a net *buyer* of claims ($b_1 < 0$), it has the incentive to decrease the price of the claims to make them cheaper, therefore increasing the return on the assets that mature at $t = 1$.

Note that an increase in utility at g_1 decreases the likelihood ratio $m_1^*(g_1)$, but it also increases the likelihood ratios at the rest of the contingencies $\hat{g}_1 \neq g_1$, so that the ratios integrate to unity. This is the reason for the appearance of η_1 instead of just μ_1 in the first-order condition (5), which accounts for the *net* shadow benefit or cost of increasing the worst-case beliefs and the respective prices. Due to our zero initial debt assumption, we have $\eta_1 = \mu_1$ in this simple two-period version of our model. This is not true in a multiperiod setting, as we see later in the analysis of the infinite horizon economy.

Furthermore, thinking about the implications for the tax rate, one can show, using the optimal wedge (9), that an *increase* in $\tilde{\xi}_1 = \sigma\eta_1 = \sigma\mu_1$ (equivalently, a decrease in debt in marginal utility terms) leads to an *increase* in consumption and labor, and, therefore, to a *reduction* in the tax rate, keeping everything else equal.[14] If the government is running a deficit for high shocks, financed by assets contingent on these shocks, and a surplus for low shocks, used to pay back state-contingent debt, then the incentive of the government is to decrease the tax rate (reducing, therefore, the pessimistic probability and the respective price of a claim) when there is a high government expenditure shock and to increase the tax rate (increasing, therefore, the pessimistic probability and the respective price of a claim) when there is a low government expenditure shock. Thus, the price manipulation motive acts in the *opposite* direction to the paternalistic (or mispricing) motive that we analyzed before.[15]

A last remark is due. Consider the optimal tax rate formula (10) in the case of constant risk aversion γ and constant elasticity of the marginal disutility of labor ϕ_h (which is equal to the inverse of the Frisch elasticity). Then, with full confidence in the model, the tax rate is *constant* among states of the world, $\tau_1 = \Phi(\gamma + \phi_h)/(1 + \Phi(1 + \phi_h))$, whereas now it *varies* at each state due to the paternalism incentive (m_1^*) and the price manipulation incentive ($\tilde{\xi}_1$). Note, furthermore, that for these particular utility functions, we can read the two basic forces that lead to an increase or decrease of the tax rate from $\tau_1 = \Phi(\gamma + \phi_h)/(1/m_1^* + \tilde{\xi}_1 + \Phi(1 + \phi_h))$.

2.2 *A three-period economy*

Add one more period, $t = 2$, and for the sake of simplicity, assume that there is no uncertainty at $t = 2$, $g_2 = \bar{g}$.[16] The shock histories in the economy are $g^2 = (g_1, \bar{g})$, with

[14] Remember from footnote 11 that $\tilde{\xi}_1$ stands for the normalized shadow value of the household's utility. An increase in $\tilde{\xi}_1$ captures the net marginal benefit of decreasing the worst-case beliefs of the household by means of increasing V_1. This comparative statics result holds for the same utility functions as in footnote 13. See the Appendix for details.

[15] If the government adopted the household's welfare criterion, the paternalism force would be absent. The two forces could also act in the same direction, depending on the strength of the government's doubts relative to the household's. See Karantounias (2011).

[16] The case with uncertainty is covered in the infinite horizon economy.

approximating probabilities $\pi_1(g_1)$. The resource constraint at $t = 2$ is

$$c_2(g_1, \bar{g}) + \bar{g} = h_2(g_1, \bar{g}), \tag{11}$$

and the preferences of the cautious household take the form

$$\min_{m_1(g_1)\geq 0} \sum_{g_1} \pi_1(g_1)m_1(g_1)\big[u(c_1(g_1)) + v(1 - h_1(g_1)) + \beta\big(u(c_2(g_1, \bar{g})) + v(1 - h_2(g_1, \bar{g}))\big)\big] + \theta \sum_{g_1} \pi_1(g_1)m_1(g_1) \ln m_1(g_1).$$

The worst-case beliefs of the household are

$$m_1^*(g_1) = \frac{\exp(-V_1(g_1)/\theta)}{\sum_{g_1} \pi_1(g_1) \exp(-V_1(g_1)/\theta)}, \tag{12}$$

where

$$V_1(g_1) = u(c_1(g_1)) + v(1 - h_1(g_1)) + \beta\big(u(c_2(g_1, \bar{g})) + v(1 - h_2(g_1, \bar{g}))\big), \tag{13}$$

i.e., the sum of period *and* discounted future utility—a manifestation of the forward-looking behavior of the household. This forward-looking element is crucial for the optimal taxation problem.

It is easy to see that the two forces that we describe in the two-period economy are present here also. Assign multipliers $\pi_1(g_1)\xi_1(g_1)$ on (13), let $\tilde{\xi}_1$ denote the normalized multiplier $\tilde{\xi}_1 \equiv \xi_1/m_1^*$, and let the rest of the multipliers be as in the two-period economy. Then the optimal wedge equation for period $t = 1$ is as in (9), with the qualification that worst-case beliefs now are formed by taking into account the discounted value of utility, as (12) shows. The marginal incentives of manipulating the price of a state-contingent claim at $t = 1$, $q_1(g_1) = \pi_1(g_1)m_1^*(g_1)u'(c_1(g_1))/\hat{\lambda}$,[17] by means of the cautious household beliefs are captured by the shadow value of utility $\tilde{\xi}_1$ and depend again on the government asset position in period $t = 1$. This now consists of *both* the current surplus or deficit *and* the present value of the future surplus or deficit,

$$\tilde{\xi}_1 = \sigma\eta_1 = \sigma\mu_1 = \sigma\Phi\big[u'(c_1)c_1 - v'(1 - h_1) + \beta(u'(c_2)c_2 - v'(1 - h_2)h_2)\big] = \sigma\Phi u'(c_1)b_1,$$

where $b_1 = \tau_1 h_1 - g_1 + (q_2/q_1)b_2$, q_2 is the price of a claim contingent on history $(g_1, \bar{g})$, $q_2(g_1, \bar{g}) = \beta\pi_1(g_1)m_1^*(g_1)u'(c_2(g_1, \bar{g}))/\hat{\lambda}$, q_2/q_1 is the inverse of the gross interest rate between period one and period two, and b_2 is the government surplus or deficit at period $t = 2$, $b_2 = \tau_2 h_2 - \bar{g}$. As before, the fiscal authority faces two opposite incentives: the desire to tax high when fiscal shocks are high, because they are considered relatively improbable, whereas at the same time, the desire to tax the very same events less if, as is typically the case, the present value of government surpluses is negative ($b_1 < 0$), so as to reduce the equilibrium price of the state-contingent claims that it buys.

[17]The variable $\hat{\lambda}$ is the multiplier on the household's intertemporal budget constraint from the household's optimization problem. It can be eliminated according to the preferred normalization of prices.

Turning to the optimal wedge at period $t=2$, we have now

$$\begin{aligned} & v'(1-h_2(g_1,\bar{g}))-u'(c_2(g_1,\bar{g})) \\ & \quad = \frac{\Phi}{1/m_1^*(g_1)+\tilde{\xi}_1(g_1)+\Phi}\big[u''(c_2(g_1,\bar{g}))c_2(g_1,\bar{g})+v''(1-h_2(g_1,\bar{g}))h_2(g_1,\bar{g})\big]. \end{aligned} \tag{14}$$

Several comments are in place. Use the resource constraint (11) to substitute for labor in (14) and remember that with full confidence in the model, we would have $m_1^*\equiv 1$ and $\tilde{\xi}_1\equiv 0$. Then the optimal wedge equation at $t=2$ determines optimal consumption (and, therefore, labor and the tax rate) solely as a function of the level of government expenditures at period $t=2$ and the multiplier Φ, $c_2=c(\bar{g};\Phi)$. This is the celebrated *history independence* result of Lucas and Stokey (1983), which renders the optimal plan effectively *static*. The only intertemporal link in this case occurs implicitly through the value of the multiplier Φ on the implementability constraint, and this by itself imparts no history dependence.

In the case of doubts about the model though, consumption depends on the past shock through m_1^* and $\tilde{\xi}_1$, $c_2=c(\bar{g},m_1^*,\tilde{\xi}_1;\Phi)$, and, therefore, the dependence on the past is due to both forces that we analyze in the two-period model. To interpret how the past matters, assume, for example, that there is a high realization of the fiscal shock at $t=1$, an event to which the household assigns high probability (high m_1^*), leading, therefore, to an inflated price of the claim contingent on history $(g_1,\bar{g})$, q_2. The paternalism force implies that after this high shock, the fiscal authority has an incentive to *keep* the tax rate at $t=2$ high, since the probability of this history is considered relatively low according to the fiscal authority. Thus, the paternalistic motive makes the tax rate *persistent*, motivating the fiscal authority to keep the tax rate high (low) *following* high (low) shocks.

Things become more interesting if we consider the price manipulation efforts through the cautious beliefs of the household in this dynamic setup. Exactly because the household is *forward-looking* in forming worst-case scenarios, the tax rate at $t=2$ affects the household's utility at $t=1$ and, therefore, affects the equilibrium price of state-contingent claims at $t=1$ and at $t=2$, q_1 and q_2, forcing the fiscal authority to take into account the past. The absence of uncertainty at $t=2$ makes the household's likelihood ratio m_1^* the relevant object of interest. The shadow value $\tilde{\xi}_1$ of affecting the household's beliefs through utility V_1 captures exactly that in (14) and indicates that the marginal incentive to affect prices *persists* over time.[18] A high shock at $t=1$, for which the fiscal authority buys assets and sets a low tax rate, motivates the fiscal authority to keep the tax rate low at $t=2$ so as to keep period utility at period $t=2$ high and, therefore, keep discounted utility at period $t=1$ high. As a result, there is an incentive to keep the tax rate low (high) *following* high (low) shocks.

[18] Note that if we assigned a second multiplier ξ_2 on the period utility at $t=2$, we would get $\tilde{\xi}_2=\tilde{\xi}_1$, where $\tilde{\xi}_2\equiv\xi_2/m_1^*$. This reflects the fact that the absence of uncertainty at $t=2$ makes the conditional distortion of beliefs for period $t=2$ identically equal to unity, $m_2^*\equiv 1$. Thus, the shadow value to the fiscal authority of the period utility at $t=2$ is just equal to the shadow value of discounted utility at $t=1$, since there is obviously no room for affecting the conditional beliefs of the household for period $t=2$. See the infinite horizon economy for the general case where this does not hold.

3. The infinite horizon economy

In this section we proceed to the full-blown infinite horizon economy. Time $t \geq 0$ is discrete and the horizon is infinite. Labor is the only input to a linear technology that produces one perishable good that can be allocated to private consumption c_t or government consumption g_t. The only source of uncertainty is an exogenous sequence of government expenditures g_t that potentially takes on a finite or countable number of values. Let $g^t = (g_0, \dots, g_t)$ denote the history of government expenditures. Equilibrium plans for work and consumption have date t components that are measurable functions of g^t. A representative agent is endowed with one unit of time, works $h_t(g^t)$, enjoys leisure $l_t(g^t) = 1 - h_t(g^t)$, and consumes $c_t(g^t)$ at history g^t for each $t \geq 0$. One unit of labor can be transformed into one unit of the good, which leads, under the competitive assumption, to a real wage $w_t(g^t) = 1$ for all $t \geq 0$ and any history g^t. Feasible allocations satisfy

$$c_t(g^t) + g_t = h_t(g^t). \tag{15}$$

The government finances its time t expenditures either by using a linear tax $\tau_t(g^t)$ on labor income or by issuing a vector of state-contingent debt $b_{t+1}(g_{t+1}, g^t)$ that is sold at price $p_t(g_{t+1}, g^t)$ at history g^t and promises to pay one unit of the consumption good if government expenditures are g_{t+1} next period and zero otherwise. The one-period government budget constraint at t is

$$b_t(g^t) + g_t = \tau_t(g^t) h_t(g^t) + \sum_{g_{t+1}} p_t(g_{t+1} | g^t) b_{t+1}(g_{t+1}, g^t).$$

Equivalently, we can work with an Arrow–Debreu formulation in which all trades occur at date 0 at Arrow–Debreu history-contingent prices $q_t(g^t)$. In this setting, the government faces the single intertemporal budget constraint

$$b_0 + \sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t) g_t \leq \sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t) \tau_t(g^t) h_t(g^t).$$

3.1 *Fear of model misspecification*

The representative agent and the government share an approximating model in the form of a sequence of joint densities $\pi_t(g^t)$ over histories g^t $\forall t \leq \infty$. Following Hansen and Sargent (2005), we characterize model misspecifications with multiplicative perturbations that are martingales with respect to the approximating model. The representative agent, but not the government, fears that the approximating model is misspecified in the sense that the history of government expenditures will actually be drawn from a joint density that differs from the approximating model but is absolutely continuous with respect to the approximating model over finite time intervals. Thus, by the Radon–

Nikodym theorem, there exists a nonnegative random variable M_t with $E(M_t) = 1$ that is a measurable function of the history g^t and that has the interpretation of a change of measure. The operator E denotes expectation with respect to the approximating model throughout the paper. The random variable M_t, which we take to be a likelihood ratio $M_t(g^t) = \tilde{\pi}_t(g^t)/\pi_t(g^t)$ of a distorted density $\tilde{\pi}_t$ to the approximating density π_t, is a martingale, i.e., $E_t M_{t+1} = M_t$. Here the tilde refers to a distorted model. Evidently, we can compute the mathematical expectation of a random variable $X_t(g^t)$ under a distorted measure as

$$\tilde{E}(X_t) = E(M_t X_t).$$

To attain a convenient decomposition of M_t, define

$$m_{t+1} \equiv \frac{M_{t+1}}{M_t} \quad \text{for } M_t > 0$$

and let $m_{t+1} \equiv 1$ when $M_t = 0$ (i.e., when the distorted model assigns zero probability to a particular history). Then

$$M_{t+1} = m_{t+1} M_t = M_0 \prod_{j=1}^{t+1} m_j.$$

The nonnegative random variable m_{t+1} distorts the conditional probability of g_{t+1} given history g^t, so that it is a conditional likelihood ratio $m_{t+1} = \tilde{\pi}_{t+1}(g_{t+1}|g^t)/\pi_{t+1}(g_{t+1}|g^t)$. It has to satisfy the restriction that $E_t m_{t+1} = 1$ to qualify as a distortion to the conditional measure. We measure discrepancies between conditional distributions by *conditional* relative entropy, which is defined as

$$\varepsilon_t(m_{t+1}) \equiv E(m_{t+1} \log m_{t+1} | g^t).$$

3.2 *Preferences*

The multiplier preferences of Hansen and Sargent (2001) and Hansen et al. (2006) in the infinite horizon economy take the form[19]

$$\min_{\{m_{t+1}, M_t\}_{t=0}^{\infty} \geq 0} \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t(g^t) U(c_t(g^t), 1 - h_t(g^t)) + \beta\theta \sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) M_t(g^t) \varepsilon_t(m_{t+1}) \tag{16}$$

[19] In effect, we constrain the set of perturbations by a constraint on a measure of discounted entropy,

$$\beta E\left[\sum_{t=0}^{\infty} \beta^t M_t E(m_{t+1} \log m_{t+1} | g^t) \Big| g_0\right] \leq \eta,$$

where η measures the size of an entropy ball of models surrounding the approximating model. This constraint could be used to formulate the constraint preferences of Hansen and Sargent (2001). They discuss the relation between constraint preferences and the multiplier preferences featured in this paper and show how to construct η ex post as a function of the multiplier θ in (16) and other parameters.

with $\theta > 0$ and $U(c_t, 1-h_t)$ satisfying the same monotonicity, concavity, and differentiability assumptions as in Section 2.[20]

3.3 *The representative household's problem*

For any sequence of random variables $\{a_t\}$, let $a \equiv \{a_t(g^t)\}_{t,g^t}$. The problem of the consumer is

$$\max_{c,h} \min_{M\geq 0, m\geq 0} \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t(g^t) \Big[U(c_t(g^t), 1-h_t(g^t)) + \theta\beta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) \ln m_{t+1}(g^{t+1}) \Big]$$

subject to

$$\sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t) c_t(g^t) \leq \sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t)(1-\tau_t(g^t)) h_t(g^t) + b_0 \tag{17}$$

$$c_t(g^t) \geq 0, \qquad h_t(g^t) \in [0,1] \quad \forall t, g^t$$

$$M_{t+1}(g^{t+1}) = m_{t+1}(g^{t+1}) M_t(g^t), \quad M_0 = 1 \; \forall t, g^t \tag{18}$$

$$\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) = 1 \quad \forall t, g^t. \tag{19}$$

We assume that uncertainty at $t=0$ is realized, so $\pi_0(g_0) = 1$. Thus, the distortion of the probability of the initial period is normalized to be unity, so that $M_0 \equiv 1$. Inequality (17) is the intertemporal budget constraint of the household. The right side is the discounted present value of after tax labor income plus an initial asset position b_0 that can assume positive (denoting government debt) or negative (denoting government assets) values.

3.4 *The inner problem: Choosing beliefs*

The inner problem chooses (M, m) to minimize the utility of the representative household subject to the law of motion of the martingale M and the restriction that the conditional distortion m integrates to unity. The optimal conditional distortion takes the exponentially twisting form[21]

$$m^*_{t+1}(g^{t+1}) = \frac{\exp(-V_{t+1}(g^{t+1})/\theta)}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp(-V_{t+1}(g^{t+1})/\theta)}, \quad \text{all } t \geq 0, g^t. \tag{20}$$

[20] The multiplier preferences can be written recursively as

$$V_t = U(c_t, 1-h_t) + \beta \min_{m_{t+1}} \{E_t m_{t+1} V_{t+1} + \theta \varepsilon_t(m_{t+1})\}.$$

[21] See the Appendix for the derivation of this formula.

where V_t is the utility of the household under the distorted measure, which follows the recursion

$$V_t = U(c_t, 1-h_t) + \beta[E_t m^*_{t+1} V_{t+1} + \theta E_t m^*_{t+1} \ln m^*_{t+1}]. \tag{21}$$

Equations (20) and (21) are the first-order conditions for the minimization problem with respect to m_{t+1} and M_t. Substituting (20) into (21) gives

$$V_t = U(c_t, 1-h_t) + \frac{\beta}{\sigma} \ln E_t(\exp(\sigma V_{t+1})), \tag{22}$$

where $\sigma \equiv -1/\theta$. Thus, the martingale distortion evolves according to

$$M^*_{t+1} = \frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp(\sigma V_{t+1}(g^{t+1}))} M^*_t, \quad M_0 \equiv 1. \tag{23}$$

Equation (23) asserts that the martingale distortion attaches higher probabilities to histories with low continuation utilities and lower probabilities to histories with high continuation utilities. Such exponential tilting of probabilities summarizes how the representative household's distrust of the approximating model produces conservative probability assessments that give rise to an indirect utility function that solves the recursion (22), an example of the discounted risk-sensitive preferences of Hansen and Sargent (1995).[22] For $\theta = \infty$ (or, equivalently, $\sigma = 0$), the conditional and unconditional distortion become unity $M^*_t = m^*_t = 1$, expressing the lack of doubts about the approximating model.

3.5 *Outer problem: Choosing* $\{c_t, h_t\}$ *plan*

An interior solution to the maximization problem of the household satisfies the intratemporal labor supply condition

$$\frac{U_l(g^t)}{U_c(g^t)} = 1 - \tau_t(g^t) \tag{24}$$

that equates the marginal rate of substitution (MRS) between consumption and leisure to the after tax wage rate and the intertemporal Euler equation

$$q_t(g^t) = \beta^t \pi_t(g^t) M^*_t(g^t) \frac{U_c(g^t)}{U_c(g_0)}. \tag{25}$$

Here we normalize the price of an Arrow–Debreu security at $t = 0$ to unity, so $q_0(g_0) \equiv 1$. The implied price of one-period state-contingent debt (an Arrow security) is

$$p_t(g_{t+1}, g^t) = \beta \pi_{t+1}(g_{t+1}|g^t) \frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp(\sigma V_{t+1}(g^{t+1}))} \frac{U_c(g^{t+1})}{U_c(g^t)}.$$

In the infinite horizon case, doubts about the model show up as a worst-case conditional density in the determination of the equilibrium price of an Arrow security. The

[22] The risk-sensitive recursion is closely related to the preferences of Epstein and Zin (1989) and Weil (1990).

stochastic discount factor under the approximating model has an additional multiplicative element that depends on the endogenous, forward-looking continuation utility.

Definition 2. A competitive equilibrium is a consumption–labor allocation (c, h), distortions to beliefs (m, M), a price system q, and a government policy (g, τ) such that (a) given (q, τ), (c, h) and (m, M) solve the household's problem, and (b) markets clear, so that $c_t(g^t) + g_t = h_t(g^t)\ \forall t, g^t$.

4. The problem of the fiscal authority

The paternalistic fiscal authority chooses at $t = 0$ a competitive equilibrium allocation that maximizes the expected utility of the representative household under the approximating model.

4.1 *Primal approach*

The fiscal authority chooses allocations subject to the resource constraint (15) and implementability constraints imposed by the competitive equilibrium.

Proposition 1. *The fiscal authority faces the implementability constraints*

$$\sum_{t=0}^{\infty}\beta^t\sum_{g^t}\pi_t(g^t)M_t^*(g^t)U_c(g^t)c_t(g^t) = \sum_{t=0}^{\infty}\beta^t\sum_{g^t}\pi_t(g^t)M_t^*(g^t)U_l(g^t)h_t(g^t) + U_c(g_0)b_0, \tag{26}$$

the law of motion for the martingale that represents distortions to beliefs (23), and the recursion for the representative household's value function (22).

Proof. In addition to the resource constraint, the competitive equilibrium is characterized fully by the household's two Euler equations, the intertemporal budget constraint (17) that holds with equality at an optimum, and equations (23) and (22), which describe the evolution of the endogenous beliefs of the agent. Use (24) and (25) to substitute for prices and after tax wages in the intertemporal budget constraint to obtain (26). □

Definition 3. The fiscal authority's problem is

$$\max_{(c,h,M^*,V)}\sum_{t=0}^{\infty}\beta^t\sum_{g^t}\pi_t(g^t)U(c_t(g^t), 1-h_t(g^t))$$

subject to

$$\sum_{t=0}^{\infty}\beta^t\sum_{g^t}\pi_t(g^t)M_t^*(g^t)[U_c(g^t)c_t(g^t) - U_l(g^t)h_t(g^t)] = U_c(g_0)b_0 \tag{27}$$

$$c_t(g^t) + g_t = h_t(g^t) \quad \forall t, g^t \tag{28}$$

$$M^*_{t+1}(g^{t+1}) = \frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)\exp(\sigma V_{t+1}(g^{t+1}))} M^*_t(g^t), \quad M_0(g_0) = 1\ \forall t, g^t \tag{29}$$

$$\begin{aligned} V_t(g^t) &= U(c_t(g^t), 1-h_t(g^t)) \\ &\quad + \frac{\beta}{\sigma}\ln\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)\exp(\sigma V_{t+1}(g^{t+1})) \quad \forall t, g^t, t \geq 1. \end{aligned} \tag{30}$$

In contrast to the two-period economy, the fiscal authority has to take into account how the worst-case beliefs of the household evolve and, therefore, it needs to keep track of the law of motion of M^*_t, (29). The increments to the endogenous likelihood ratio M^*_t are determined by the household's utility V_t, which necessitates the addition of the utility recursion (30)—a promise-keeping constraint—to the implementability constraints of the problem. Note that we could interpret the minimization problem of the household in the description of the preferences in (16) as the problem of a malevolent alter ego who, by choosing a worst-case probability distortion, motivates the household to value robust decision rules. Along the lines of this interpretation, the policy problem becomes a Stackelberg game with one leader and *two* followers, namely, the representative household's maximizing self and the representative household's malevolent alter ego.

4.2 *First-best benchmark (i.e., lump-sum taxes available)*

By first-best, we mean the allocation that maximizes the expected utility of the household under π subject to the resource constraint (15). Note that for *any* beliefs of the fiscal authority, the first-best is characterized by the condition $U_l(g^t)/U_c(g^t) = 1$ and the resource constraint (15), so the first-best allocation $(\hat{c}, \hat{h})$ is independent of probabilities π. Private sector beliefs affect asset prices through (25), but not the allocation. Because lump-sum taxes are not available in our model, the fiscal authority's and the household's beliefs both affect allocations.

4.3 *Optimality conditions of the government's problem*

For convenience, define $\Omega(c_t(g^t), h_t(g^t)) \equiv U_c(g^t)c_t(g^t) - U_l(g^t)h_t(g^t)$. Note that Ω_t represents the equilibrium government surplus or deficit in marginal utility terms, $\Omega_t = U_{ct}[\tau_t h_t - g_t]$. Attach multipliers Φ, $\beta^t\pi_t(g^t)\lambda_t(g^t)$, $\beta^{t+1}\pi_{t+1}(g^{t+1})\mu_{t+1}(g^{t+1})$, and $\beta^t\pi_t(g^t)\xi_t(g^t)$ to constraints (27), (28), (29), and (30), respectively.

First-order necessary conditions[23] for an interior solution are

$$c_t, t \geq 1: \quad U_c(g^t) + \xi_t(g^t)U_c(g^t) + \Phi M^*_t(g^t)\Omega_c(g^t) = \lambda_t(g^t) \tag{31}$$

$$h_t, t \geq 1: \quad -U_l(g^t) - \xi_t(g^t)U_l(g^t) + \Phi M^*_t(g^t)\Omega_h(g^t) = -\lambda_t(g^t) \tag{32}$$

$$M^*_t, t \geq 1: \quad \mu_t(g^t) = \Phi\Omega(g^t) + \beta\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)m^*_{t+1}(g^{t+1})\mu_{t+1}(g^{t+1}) \tag{33}$$

[23]We set up the Lagrangian of the policy problem and derive the first-order conditions in the Appendix.

$$V_t,\, t \geq 1: \quad \xi_t(g^t) = \sigma m_t^*(g^t) M_{t-1}^*(g^{t-1}) \Big[\mu_t(g^t) - \sum_{g_t} \pi_t(g_t|g^{t-1}) m_t^*(g^t) \mu_t(g^t) \Big] + m_t^*(g^t) \xi_{t-1}(g^{t-1}) \tag{34}$$

$$c_0: \quad U_c(g_0) + \xi_0 U_c(g_0) + \Phi M_0 \Omega_c(g_0) = \lambda_0(g_0) + \Phi U_{cc}(g_0) b_0 \tag{35}$$

$$h_0: \quad -U_l(g_0) - \xi_0 U_l(g_0) + \Phi M_0 \Omega_h(g_0) = -\lambda_0(g_0) - \Phi U_{cl}(g_0) b_0. \tag{36}$$

In (33) and (34), we use expression (20) for the optimal conditional likelihood ratio m_{t+1}^* to save notation.

Two remarks are in order. In formulating the taxation problem, the last constraint (22) applies only from period one on since the value of the agent at $t = 0$, V_0, is not relevant to the problem due to the normalization $M_0 \equiv 1$. We can set the initial value of the multiplier equal to zero, $\xi_0 = 0$, to accommodate this. Equivalently, we could maximize with respect to V_0 to get an additional first-order condition $\xi_0 = 0$. Furthermore, since $\xi_0 = 0$, $M_0 = 1$, the first-order conditions (35) and (36) for the initial consumption–labor allocation (c_0, h_0) are the same as the respective initial period first-order conditions for the special Lucas and Stokey (1983) case where the representative consumer fears no misspecification.

The first-order conditions (31)–(36) together with the constraints (27)–(30) determine the optimal plan.

5. Characterizing the optimal plan

5.1 *Optimal wedge*

Substituting the derivatives of Ω with respect to c and h into first-order conditions (31) and (32), and combining the resulting expressions to eliminate the shadow value of output λ_t delivers an expression for the optimal wedge for $t \geq 1$,[24] which, in terms of the normalized multiplier $\tilde{\xi}_t \equiv \xi_t / M_t^*$, $\tilde{\xi}_0 \equiv 0$, takes the form[25]

$$U_l(g^t) - U_c(g^t) = \frac{\Phi}{1/M_t^*(g^t) + \tilde{\xi}_t(g^t) + \Phi} \times \big[U_{cc}(g^t) c_t(g^t) - U_{cl}(g^t)(c_t(g^t) + h_t(g^t)) + U_{ll}(g^t) h_t(g^t) \big]. \tag{37}$$

The corresponding optimal tax rate is

$$\tau_t = \frac{\Phi}{1/M_t^* + \tilde{\xi}_t + \Phi(1 + \epsilon_{h,t})} \Big[\gamma_{\mathrm{RA},t} + \frac{U_{cl}}{U_c}(c_t + h_t) + \epsilon_{h,t} \Big], \quad t \geq 1,$$

[24] The optimal wedge at the initial period is

$$U_l(g_0) - U_c(g_0) = \frac{\Phi}{1 + \Phi} [U_{cc}(g_0)(c_0 - b_0) - U_{cl}(g_0)(c_0 - b_0 + h_0) + U_{ll}(g_0) h_0].$$

In the absence of initial debt $b_0 = 0$, the optimal wedge at $t = 0$ is determined by (37) for $(M_0, \tilde{\xi}_0) = (1, 0)$. Initial consumption is a function of (g_0, b_0) and Φ, $c_0 = c(g_0, b_0; \Phi)$.

[25] This normalization amounts essentially to multiplying the household's utility recursion (30) with M_t^* and assigning the multiplier $\tilde{\xi}_t$ on that constraint.

where $\gamma_{\mathrm{RA},t}$ and $\epsilon_{h,t}$ stand again for the coefficient of relative risk aversion and the elasticity of the marginal disutility of labor. A sufficient condition for a positive tax rate is $U_{cl} \geq 0$, following the same arguments as in Section 2.

The optimal wedge formula (37) generalizes the two forces that we identify in the two-period economy: the paternalistic motive of the fiscal authority, captured by M_t^*, and the price manipulation through the household's cautious beliefs, captured by the multiplier $\tilde{\xi}_t$, which measures the shadow value to the fiscal authority of the representative household's utility.

5.2 *Paternalistic motives in infinite horizon*

In the two-period economy, the paternalistic motive was captured by the conditional likelihood ratio m_1^* (which equates trivially the unconditional likelihood ratio in that setup). In the infinite horizon economy, the optimal wedge equation (37) shows that this role is played, in contrast, by the *unconditional* likelihood ratio M_t^*, which by construction consists of the product of the conditional likelihood ratios

$$M_t^* = m_t^* M_{t-1}^*,$$

generalizing naturally our two-period insights. Using (37), we can show that similarly to Section 2, an increase in M_t^*, which corresponds to a *history* of shocks that the fiscal authority does not consider very probable, leads to an increased tax rate, keeping everything else equal.[26] The opposite situation happens for histories that the fiscal authority considers more probable relative to the household.

The decomposition of the unconditional likelihood ratio in terms of *all* past conditional likelihood ratios m_t^* is helpful for understanding the paternalistic motive. Each conditional distortion m_t^* depends on V_t, as shown in (20). A sequence of low utility events from $t = 1$ until the current period leads to high increments m_t^* over time and, therefore, to an increasing likelihood ratio M_t^*.[27] Therefore, a sequence of high government expenditure shocks, which we typically associate with low utility events, leads to an increasing sequence of M_t^* and, therefore, to an *increasing* tax rate over time (which is associated with an increasing sequence of debt positions or a decreasing sequence of asset positions of the government). At first, note that this is an indication of *persistence* of the tax rate due to the paternalistic motive as we also see in the three-period economy. Furthermore, we note that the paternalistic motive implies a *back-loading* of taxes in the case of a sequence of high shocks (whereas without doubts about the model, the tax rate remains constant) due to the increasing implausibility of these histories in the eyes of the fiscal authority.

5.3 *Manipulation of expectations and prices in infinite horizon*

Turning now to the price manipulation motives of the government, consider the first-order condition with respect to V_t, (34), which determines the evolution of ξ_t and

[26] See the Appendix for the comparative statics results with respect to $(M^*, \tilde{\xi})$.

[27] In the three-period economy, we have $M_2^* = m_1^*$, since there is no uncertainty at period $t = 2$ and, therefore, $m_2^* \equiv 1$.

therefore of $\tilde{\xi}_t$,

$$\xi_t = \sigma m_t^* M_{t-1}^* \eta_t + m_t^* \xi_{t-1}, \quad t \geq 1, \xi_0 = 0, \tag{38}$$

where

$$\eta_t \equiv \mu_t - E_{t-1} m_t^* \mu_t$$

stands now for the *conditional* innovation in μ_t under the household's distorted measure, with $E_{t-1} m_t^* \eta_t = E_{t-1} m_t^* \mu_t - E_{t-1} m_t^* \cdot E_{t-1} m_t^* \mu_t = 0$, since $E_{t-1} m_t^* = 1$.

The corresponding law of motion in terms of the normalized multiplier $\tilde{\xi}_t$ becomes

$$\tilde{\xi}_t = \sigma \eta_t + \tilde{\xi}_{t-1}, \quad t \geq 1, \tilde{\xi}_0 = 0. \tag{39}$$

The multiplier μ_t, which captures the shadow value to the fiscal authority of increasing the likelihood ratio M_t^*, can be found by iterating forward the first-order condition with respect to M_t^* ((33)), which, by remembering that $\Omega(g^t)$ stands for the government surplus in marginal utility terms, delivers

$$\mu_t(g^t) = \Phi U_c(g^t) \sum_{i=0}^{\infty} \sum_{g^{t+i}|g^t} q_{t+i}^t(g^{t+i})[\tau_{t+i}(g^{t+i}) h_{t+i}(g^{t+i}) - g_{t+i}],$$

where

$$q_{t+i}^t(g^{t+i}) \equiv \frac{q_{t+i}(g^{t+i})}{q_t(g^t)} = \beta^i \pi_{t+i}(g^{t+i}|g^t) \prod_{j=1}^{i} m_{t+j}^*(g^{t+j}) \frac{U_c(g^{t+i})}{U_c(g^t)},$$

the equilibrium price of an Arrow–Debreu security in terms of consumption at history g^t. Thus, using the intertemporal budget constraint at time t allows us to rewrite μ_t as $\mu_t = \Phi U_{ct} b_t$ and interpret η_t as the innovation in debt in marginal utility terms multiplied by the cost of distortionary taxation Φ, $\eta_t = \Phi[U_{ct} b_t - E_{t-1} m_t^* U_{ct} b_t]$.

Analyzing now the price manipulation motives in the infinite horizon economy, note first that, as in Section 2, an increase in $\tilde{\xi}_t$ leads to a decrease in the tax rate, keeping everything else equal. However, in contrast to the two-period case, $\tilde{\xi}_t$ depends now on the *cumulative* innovation in debt in marginal utility terms,

$$\tilde{\xi}_t = \sigma H_t,$$

where $H_t \equiv \sum_{i=1}^t \eta_i$ and $H_0 \equiv 0$, indicating that all *past* innovations in debt η_i matter for the decisions of the fiscal authority. The intuition behind this result is a generalization of the intuition we highlighted in the three-period economy. It helps to write down the equilibrium intertemporal budget constraint:

$$U_{c0} b_0 = \Omega_0 + \beta E_0 M_1^* \Omega_1 + \cdots + \beta^{t-1} E_0 M_{t-1}^* \Omega_{t-1} + \underbrace{\beta^t E_0 M_t^* U_{ct} b_t}_{U_{c0} \sum_{g^t} q_t(g^t) b_t(g^t)}. \tag{40}$$

The fiscal authority has an incentive to decrease the price (by decreasing the tax rate) of a history-contingent claim $q_t = \beta^t \pi_t M_t^* U_{ct}/U_{c0}$ by means of the endogenous worst-case beliefs of the household in situations where it ex ante buys assets relative to the value of the government portfolio ($\eta_t < 0$), and to increase the price (by increasing the tax rate) when it sells debt relative to the value of the government portfolio ($\eta_t > 0$).[28] Both of these actions relax the constraint (40) in the relevant contingencies.

This is not the whole story, though. The past innovations in debt matter due to the forward-looking nature of the worst-case beliefs of the household. Any change in $V_t(g^t)$ through the tax rate τ_t affects all past continuation utilities $\{V_1(g^1), \ldots, V_{t-1}(g^{t-1})\}$ along the history g^t through recursion (22) and, therefore, all likelihood ratios M_i^*, $i = 1, \ldots, t$. As a result, *all* equilibrium prices q_i, $i = 1, \ldots, t$, along this history are affected. This is why the normalized shadow value of utility $\tilde{\xi}_t$ consists of the cumulative innovation H_t, tracking dates in the past that the government was lending or borrowing and the corresponding marginal incentives to affect equilibrium prices. The cumulative innovation H_t captures the essence of *commitment* to the household's utility recursion and to the corresponding evolution of the endogenous worst-case beliefs: the fiscal authority must take into account how a g^t-contingent action chosen at $t = 0$ affects the choices of the forward-looking household and equilibrium asset prices along the history g^t.

Furthermore, if the government hedges government expenditure shocks by buying assets contingent on high shocks and selling debt contingent on low shocks, we would have a taxation incentive that acts in the opposite direction to the paternalistic motive, as in the simpler economies that we examined earlier. Thus, a sequence of high expenditure shocks that leads to a sequence of negative innovations η_t, lead to a decreasing tax rate over time.[29] We can think of this as a tax *front-loading* incentive in the face of a sequence of high shocks, so as to reduce asset prices properly along this shock history.

5.4 *Smoothing*

It is interesting to note that a novel intertemporal smoothing motive emerges when we consider the price manipulation efforts of the fiscal authority. The government exhibits a desire to *smooth* the shadow value of the household's utility ξ_t—essentially the shadow value of the household's worst-case beliefs—by making it a *martingale* according to the government's beliefs π_t. Thus, the best forecast of the future value of the price manipulation motive is its current value, which is *not* equal to zero, in contrast to the full confidence economy.

Proposition 2 (Smoothing). *The multiplier ξ_t is a martingale under the approximating model π_t. The normalized multiplier $\tilde{\xi}_t$ is a martingale with respect to household's worst-case beliefs $\pi_t \cdot M_t^*$.*

[28] Remember that the innovation η_t captures the *net* effect marginal benefit or cost of affecting V_t, due to the fact that conditional distortions are interconnected among states. The innovation η_1 is also the relevant object in the two- and three-period economy, but it is reducing just to $\mu_1 = \Phi u'(c_1) b_1$, since the present value of government surpluses was zero in these two economies. For simplicity, we are going to refer to $\eta_t < 0$ and $\eta_t > 0$ as assets and debt, respectively.

[29] It is obvious from the law of motion (39) that a negative innovation $\eta_t < 0$ leads to an *increase* in $\tilde{\xi}_t$ (and, therefore, an incentive to set the tax rate lower over time), $\tilde{\xi}_t > \tilde{\xi}_{t-1}$, whereas a positive innovation $\eta_t > 0$ leads to a *decrease* (and, therefore, an incentive to set the tax rate higher over time), $\tilde{\xi}_t < \tilde{\xi}_{t-1}$.

Proof. Taking conditional expectation with respect to the approximating model π given history g^{t-1} in the law of motion (38) for ξ_t and remembering that variables dated at t are measurable functions of the history g^t, we get

$$\begin{aligned} E_{t-1}\xi_t &= \sigma M^*_{t-1}E_{t-1}m^*_t\eta_t + \xi_{t-1}E_{t-1}m^*_t \\ &= \xi_{t-1}, \end{aligned}$$

since $E_{t-1}m^*_t\eta_t = 0$ and $E_{t-1}m^*_t = 1$. We can take conditional expectations in the law of motion of $\tilde{\xi}_t$ ((39)) and repeat the same steps to show that $E_{t-1}m^*_t\tilde{\xi}_t = \tilde{\xi}_{t-1}$. Or, even simpler, given that $\xi_t = M^*_t\tilde{\xi}_t$ and that $E_{t-1}\xi_t = \xi_{t-1}$, we have $E_{t-1}M^*_t\tilde{\xi}_t = M^*_{t-1}E_{t-1}m^*_t\tilde{\xi}_t = \xi_{t-1}$, so $E_{t-1}m^*_t\tilde{\xi}_t = \xi_{t-1}/M^*_{t-1} \equiv \tilde{\xi}_{t-1}$. An immediate corollary of these martingale properties is that the mean value of ξ_t according to the approximating model is zero since $E(\xi_t) = E(E_0\xi_t) = E(\xi_0) = 0$ and, similarly, the mean value of $\tilde{\xi}_t$ according to the household's worst-case beliefs is zero. □

5.5 *State variables*

As is clear from the preceding analysis and the analysis in the three-period economy, the optimal plan is history-dependent due to both the paternalistic and the expectation management motive, in contrast to the Lucas and Stokey plan, where consumption is solely a function of the current shock g_t, $c^{\mathrm{LS}}_t(g^t) = c(g_t; \Phi)$. This can be readily seen from the optimal wedge (37) and the resource constraint (15) for $t \geq 1$, which delivers $c_t = c(g_t, M^*_t, \tilde{\xi}_t; \Phi)$ (implying $h_t = h(g_t, M^*_t, \tilde{\xi}_t; \Phi)$ and $\tau_t = \tau(g_t, M^*_t, \tilde{\xi}_t; \Phi)$).[30] Therefore, the allocation and taxes at t depend on the history of shocks as intermediated through M^*_t and $\tilde{\xi}_t$. The dependence of $(M^*_t, \tilde{\xi}_t)$ on the past is not degenerate since these two variables follow laws of motion (29) and (39), respectively.

The above analysis is based on the insights arising from the optimal wedge (37). We would like to know if the martingales $\tilde{\xi}_t$ (ξ_t) and M^*_t, which induce persistence to the optimal plan and capture the two forces of our model, are *sufficient* to capture the effect of history, i.e., if they can serve as *state* variables in a recursive formulation of the government's problem. We pursue this task along the lines of Marcet and Marimon (2011).

Proposition 3. *Let the approximating model of government expenditures be Markov. Then the fiscal authority's problem from period one onward can be represented recursively by keeping as a state variable the vector* (g_t, M^*_t, ξ_t) *with initial value* $(g_0, 1, 0)$. *A similar recursive formulation can be achieved in terms of* $(g_t, M^*_t, \tilde{\xi}_t)$, *with initial value* $(g_0, 1, 0)$.

See the Appendix for the proof.

To conclude, the logic of the Marcet and Marimon (2011) method (and in fact of any method that tries to represent commitment problems recursively) is to augment the state space appropriately so as to capture the restrictions that are implied by the forward-looking behavior of the household. The multiplier ξ_t (the *co-state* variable) on

[30] Note that we could achieve the same result by working with the nonnormalized multiplier ξ_t to get $c_t = c(g_t, M^*_t, \xi_t; \Phi)$.

the forward-looking implementability constraint (30) becomes a state variable, with initial value zero, which reflects the fact that the government at period one is not constrained to commit to the shadow value of its utility promises to the household, whereas the likelihood ratio M_t^* with law of motion (23) tracks the worst-case beliefs of the household, helping the identification of situations that the household considers more or less likely than the government. This augmented state allows us to express the policy problem as a functional saddle-point problem.

6. Concluding remarks

In this paper, we analyze the design of optimal fiscal policy in an environment where a government that completely trusts the probability model of exogenous government expenditures faces a public that expresses doubts about it and forms pessimistic expectations. We use a decision-theoretic model to make sense of pessimistic expectations and analyze the channels through which they affect the allocation of tax distortions over histories of shocks.

We find that a paternalistic fiscal authority that needs to resort to distortionary taxation to finance government expenditures has, on the one hand, an incentive to exploit the mispricing of the household by taxing more events that it considers unlikely relative to the household and, on the other hand, an incentive to affect equilibrium prices by managing the endogenous household's expectations about the exogenous shocks in the economy. This type of expectation management is absent in the rational expectations literature.

What lessons does our approach to modeling expectations management offer for fiscal policy? Fundamentally, the fiscal authority should shift expectations so as to lower the cost of issuing debt contingent on future—typically favorable—states of the world, that is paid back by future government surpluses. It can do this by making households think that these states are more likely to materialize. Since we model the households as endogenously pessimistic, getting them to believe these states are more likely involves making households worse off in those states by taxing them more. The reverse logic of a smaller tax on households applies for future—typically adverse—states of the world for which the government buys assets to use for financing future deficits. Thus, one implication of our model is that the fiscal authority, in its effort to increase the value of the portfolio of government securities so as to reduce the cost of distortionary taxation, is trying to curb the fears of the households by setting higher tax rates for favorable shocks and lower tax rates for adverse shocks.

We think that the intertemporal links introduced by forward-looking pessimistic households can also play an important role in other optimal policy settings, as in monetary policy or in optimal capital taxation.

Appendix A: Optimal wedge comparative statics

The derivations of the first-order conditions for the two- and three-period economy are subsumed in the infinite horizon economy and are not repeated here. Consider the comparative statics that we perform by using the optimal wedge (9) and the resource constraint (1), which are repeated here for convenience:

$$(v'(1-h)-u'(c))(1+m^*\tilde{\xi}+\Phi m^*)=\Phi m^*[u''(c)c+v''(1-h)h]$$
$$c+g=h.$$

Given g and Φ, this system of equations implicitly defines consumption and labor as functions of m^* and $\tilde{\xi}$: $c=c(m^*,\tilde{\xi})$ and $h=h(m^*,\tilde{\xi})$. We sign the partial derivatives of these functions. Note at first that the resource constraint immediately implies that $c_i=h_i$, $i=\tilde{\xi}, m^*$, where the subscript denotes the partial derivative. Implicitly differentiating the optimal wedge equation with respect to m^* delivers

$$c_{m^*}=h_{m^*}=\frac{(v'(1-h)-u'(c))(\tilde{\xi}+\Phi)-\Phi[u''(c)c+v''(1-h)h]}{K},$$

where

$$K\equiv(u''(c)+v''(1-h))(1+m^*\tilde{\xi}+2\Phi m^*)+\Phi m^*[u'''(c)c-v'''(1-h)h].$$

The numerator of c_{m^*} can be further simplified by using the optimal wedge equation to finally get

$$c_{m^*}=h_{m^*}=\frac{(u'(c)-v'(1-h))/m^*}{K}.$$

Similarly, implicitly differentiating with respect to $\tilde{\xi}$ delivers

$$c_{\tilde{\xi}}=h_{\tilde{\xi}}=\frac{m^*(v'(1-h)-u'(c))}{K}.$$

As we showed in the main text, $u'>v'$ (which implies a positive tax rate). We work under the assumption that $K<0$. Then $c_{m^*}=h_{m^*}<0$ and $c_{\tilde{\xi}}=h_{\tilde{\xi}}>0$, as claimed in the text. Furthermore, we can express the tax rate as a function of $(m^*,\tilde{\xi})$, $\tau(m^*,\tilde{\xi})=1-v'(1-h(m^*,\tilde{\xi}))/u'(c(m^*,\tilde{\xi}))$. Differentiating with respect to m^* and $\tilde{\xi}$ delivers

$$\tau_i=\frac{u''(c)v'(1-h)+v''(1-h)u'(c)}{(u'(c))^2}c_i,\quad i=m^*,\tilde{\xi}.$$

Thus, since $c_{m^*}<0$ and $c_{\tilde{\xi}}>0$, we have $\tau_{m^*}>0$ and $\tau_{\tilde{\xi}}<0$.

Sign of K. It is convenient to decompose K as

$$K=K_c+K_h,$$

where

$$K_c\equiv u''(c)(1+m^*\tilde{\xi}+2\Phi m^*)+\Phi m^*u'''(c)c$$
$$K_h\equiv v''(1-h)(1+m^*\tilde{\xi}+2\Phi m^*)-\Phi m^*v'''(1-h)h.$$

We show that $K<0$ for a power utility function of consumption, $u(c)=(c^{1-\rho}-1)/(1-\rho)$, and *either* convex marginal utility of leisure ($v'''>0$) *or* constant Frisch elasticity,

$v(1-h) = -a_h h^{1+\phi_h}/(1+\phi_h)$. Consider first K_c, which becomes

$$K_c = -\rho c^{-\rho-1}(1 + m^*\tilde{\xi} + \Phi m^*(1-\rho)).$$

Note though that for this utility function, the first-order condition of the policy problem with respect to consumption takes the form

$$1 + m^*\tilde{\xi} + \Phi m^*(1-\rho) = \lambda c^{\rho} > 0.$$

Therefore, $K_c < 0$. Furthermore, if $v''' > 0$, then $K_h < 0$, since $1 + m^*\tilde{\xi} + \Phi m^* > 0$, as shown in footnote 12. Thus, $K = K_c + K_h < 0$.

Consider now the case of constant Frisch elasticity, for which the third derivative is not positive, unless $\phi_h > 1$, since $v'''(1-h) = a_h\phi_h(\phi_h - 1)h^{\phi_h - 2}$. However, K_h becomes

$$K_h = -a_h\phi_h h^{\phi_h-1}[1 + m^*\tilde{\xi} + \Phi m^*(1+\phi_h)] < 0,$$

which again delivers the desired sign of K.

Nonseparable case. In the infinite horizon economy, we also treat the nonseparable case. Obviously, our comparative statics results for the separable case also hold there by considering the derivative of consumption (labor) with respect to M^* and $\tilde{\xi}$ (which now captures the cumulative innovation in debt). Implicitly differentiating the optimal wedge equation for nonseparable utility functions (37) and the resource constraint with respect to $(M^*, \tilde{\xi})$ delivers

$$c_{M^*} = h_{M^*} = \frac{(U_c - U_l)/M^*}{K_{\text{non}}}$$

$$c_{\tilde{\xi}} = h_{\tilde{\xi}} = \frac{M^*(U_l - U_c)}{K_{\text{non}}},$$

where K_{non} is the corresponding expression for the *nonseparable* case:

$$K_{\text{non}} \equiv (U_{cc} - 2U_{cl} + U_{ll})(1 + M^*\tilde{\xi} + 2\Phi M^*) + \Phi M^*[U_{ccc}c - U_{ccl}(2c+h) + U_{cll}(c+2h) - U_{lll}h].$$

Again, we assume that our utility functions are such that $K_{\text{non}} < 0$. If there is a positive tax rate (a sufficient condition for that is $U_{cl} \geq 0$), then $U_c > U_l$ and, therefore, $c_{M^*} = h_{M^*} < 0$ and $c_{\tilde{\xi}} = h_{\tilde{\xi}} > 0$. The tax rate derivatives in the nonseparable case are

$$\tau_i = \frac{U_{cc}U_l + U_{ll}U_c - U_{cl}(U_c + U_l)}{U_c^2}c_i, \quad i = M^*, \tilde{\xi}. \tag{41}$$

Under $U_{cl} \geq 0$, we have $c_{M^*} < 0$ and $c_{\tilde{\xi}} > 0$, and the term that multiplies the consumption derivatives c_i in (41) is negative. Therefore, $\tau_{M^*} > 0$ and $\tau_{\tilde{\xi}} < 0$.

We need to further discuss the negative sign of K_{non} in the nonseparable case. Note that when we turn off the doubts of the household by setting $\sigma = 0$, we get $(M^*, \tilde{\xi}) = (1, 0)$. Thus, K_{non} at $(1, 0)$ becomes $K_{\text{non}}(1, 0) = (U_{cc} - 2U_{cl} + U_{ll})(1 + 2\Phi) + \Phi[U_{ccc}c - U_{ccl}(2c+h) + U_{cll}(c+2h) - U_{lll}h]$. This is the second derivative of the Lagrangian with respect to consumption (after substituting for labor through the resource constraint) of the Lucas and Stokey (1983) problem for the proper value of Φ. In that case, $K_{\text{non}}(1,0) < 0$ imposes local concavity of the Lagrangian with respect to the control variables, therefore, satisfying the sufficient second-order conditions of the policy problem with full confidence in the model. Therefore, for *small* doubts about the model and a Φ close enough to the cost of distortionary taxation of Lucas and Stokey, we can justify $K_{\text{non}} < 0$ as a sufficient condition for the satisfaction of the second-order conditions of the full confidence problem. Obviously, the same argument can be made for the separable case. Note though that for the utility functions that we used before (power in consumption and convex marginal utility of leisure or constant Frisch), we showed that $K < 0$ for *any* doubts about the model.

Appendix B: Household's inner problem and optimality conditions of the fiscal authority's problem

B.1 *Inner problem in Section 3.4*

Assign multipliers $\beta^{t+1}\pi_{t+1}(g^{t+1})\rho_{t+1}(g^{t+1})$ and $\beta^t\pi_t(g^t)\nu_t(g^t)$ on constraints (18) and (19), respectively, and remember that $M_0 \equiv 1$ and $\pi_0(g_0) = 1$. Form the Lagrangian

$$L = \sum_{t=0}^{\infty}\sum_{g^t}\beta^t\pi_t(g^t)\bigg\{M_t(g^t)\bigg[U_t(g^t) + \theta\beta\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\ln m_{t+1}(g^{t+1})\bigg]$$
$$- \sum_{g_{t+1}}\beta\pi_{t+1}(g_{t+1}|g^t)\rho_{t+1}(g^{t+1})[M_{t+1}(g^{t+1}) - m_{t+1}(g^{t+1})M_t(g^t)]$$
$$- \nu_t(g^t)\bigg[\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1}) - 1\bigg]\bigg\}.$$

First-order necessary conditions for an interior solution are

$$m_{t+1}(g^{t+1}),\ t\geq 0{:}\quad \nu_t(g^t) = \beta\theta M_t(g^t)[1+\ln m_{t+1}(g^{t+1})] + \beta\rho_{t+1}(g^{t+1})M_t(g^t) \tag{42}$$

$$M_t(g^t),\ t\geq 1{:}\quad \rho_t(g^t) = U_t(g^t) + \beta\bigg[\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\rho_{t+1}(g^{t+1}) + \theta\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\ln m_{t+1}(g^{t+1})\bigg]. \tag{43}$$

The above conditions can be simplified as follows. Rearrange (42) to get

$$\ln m_{t+1}(g^{t+1}) = -\frac{\rho_{t+1}(g^{t+1})}{\theta} + \left(\frac{\nu_t(g^t)}{\beta\theta M_t(g^t)} - 1\right)$$

or

$$m_{t+1}(g^{t+1}) = \exp\left(-\frac{\rho_{t+1}(g^{t+1})}{\theta}\right)\exp\left(\frac{\nu_t(g^t)}{\beta\theta M_t(g^t)} - 1\right).$$

Taking conditional expectation of m_{t+1} and using (19) allows us to eliminate $\nu_t(g^t)$ and get

$$m^*_{t+1}(g^{t+1}) = \frac{\exp(-\rho^*_{t+1}(g^{t+1})/\theta)}{\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)\exp(-\rho^*_{t+1}(g^{t+1})/\theta)}, \tag{44}$$

where the asterisks denote optimal values. Furthermore, solving forward (43) and imposing the transversality condition $\lim_{k\to\infty}\beta^k E_t M^*_{t+k}\rho^*_{t+k}=0$ delivers

$$\begin{aligned}\rho^*_t(g^t) = {} & \sum_{i=0}^{\infty}\sum_{g^{t+i}|g^t}\beta^i\pi_{t+i}(g^{t+i}|g^t)\frac{M^*_{t+i}(g^{t+i})}{M^*_t(g^t)}\\ & \times\Bigg[U(g^{t+i})\\ & +\beta\theta\sum_{g_{t+i+1}|g^{t+i}}\pi_{t+i+1}(g_{t+i+1}|g^{t+i})m^*_{t+i+1}(g^{t+i+1})\ln m^*_{t+i+1}(g^{t+i+1})\Bigg],\quad t\geq 1.\end{aligned}$$

As is clear from the above condition, $\rho^*_t(g^t)$ represents the household's utility at history g^t, $\rho^*_t(g^t)=V_t(g^t)$. This fact, together with recursion (43) and the formula for the optimal conditional distortion (44), deliver the conditions in the text.

B.2 *First-order conditions of the policy problem*

The Lagrangian of the policy problem is

$$\begin{aligned}L = \sum_{t=0}^{\infty}\sum_{g^t}\beta^t\pi_t(g^t)\Bigg\{ & U(c_t(g^t),1-h_t(g^t))+\Phi M^*_t(g^t)\Omega(c_t(g^t),h_t(g^t))\\ & -\lambda_t(g^t)[c_t(g^t)+g_t-h_t(g^t)]-\sum_{g_{t+1}}\beta\pi_{t+1}(g_{t+1}|g^t)\mu_{t+1}(g^{t+1})\\ & \times\left[M^*_{t+1}(g^{t+1})-\frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)\exp(\sigma V_{t+1}(g^{t+1}))}M^*_t(g^t)\right]\\ & -\xi_t(g^t)\Bigg[V_t(g^t)-U(c_t(g^t),1-h_t(g^t))\\ & -\frac{\beta}{\sigma}\ln\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)\exp(\sigma V_{t+1}(g^{t+1}))\Bigg]\Bigg\}\\ & -\Phi U_c(c_0,1-h_0)b_0,\end{aligned}$$

with $\xi_0=0$, $M_0=1$, and g_0 given.

Apart from first-order condition (34), the rest of the first-order conditions of the government's maximization problem can be derived in a straightforward fashion. Now differentiate the Lagrangian with respect to $V_t(g^t)$ to get

$$\begin{aligned}V_t,\ t\geq 1:\quad & \pi_t(g_t|g^{t-1})\xi_t(g^t)\\ & = M^*_{t-1}(g^{t-1})\frac{\partial}{\partial V_t(g^t)}\left\{\frac{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))\mu_t(g^t)}{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))}\right\}\\ & \quad +\frac{\xi_{t-1}}{\sigma}\frac{\partial}{\partial V_t(g^t)}\left\{\ln\sum_{g_{t+1}}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))\right\}.\end{aligned}$$

Note that

$$\frac{\partial}{\partial V_t(g^t)}\left\{\frac{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))\mu_t(g^t)}{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))}\right\}$$

$$= \pi_t(g_t|g^{t-1})\sigma\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))}$$

$$\times\left[\mu_t(g^t) - \sum_{g_t}\pi_t(g_t|g^{t-1})\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))}\mu_t(g^t)\right]$$

$$= \pi_t(g_t|g^{t-1})\sigma m_t^*(g^t)\left[\mu_t(g_t) - \sum_{g_t}\pi_t(g_t|g^{t-1})m_t^*(g^t)\mu_t(g^t)\right]$$

and

$$\frac{\partial}{\partial V_t(g^t)}\left\{\ln\sum_{g_{t+1}}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))\right\} = \pi_t(g_t|g^{t-1})\sigma\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t}\pi_t(g_t|g^{t-1})\exp(\sigma V_t(g^t))}$$

$$= \pi_t(g_t|g^{t-1})\sigma m_t^*(g^t),$$

where we used formula (20) for the household's conditional distortion. Plugging the two derivatives back into the optimality condition and simplifying delivers (34) in the text.

Appendix C: Recursive formulation

First we give an expanded version of Proposition 3.

Proposition 3′. *Let the approximating model of government expenditures be Markov. Then the fiscal authority's problem from period one onward can be represented recursively by keeping as a state variable the vector* (g_t, M_t^*, ξ_t). *The likelihood ratio* M_t^* *and the multiplier* ξ_t *follow laws of motion*

$$M_t^* = M^*(g_t, g_{t-1}, M_{t-1}^*, \xi_{t-1}; \Phi)$$
$$\xi_t = \xi(g_t, g_{t-1}, M_{t-1}^*, \xi_{t-1}; \Phi),$$

with initial values $(M_0, \xi_0) = (1, 0)$. *The policy functions for consumption, household utility, and debt for* $t \geq 1$ *are*

$$c_t = c(g_t, M_t^*, \xi_t; \Phi)$$
$$V_t = V(g_t, M_t^*, \xi_t; \Phi)$$
$$b_t = b(g_t, M_t^*, \xi_t; \Phi).$$

A similar recursive formulation can be achieved in terms of $(g_t, M_t^*, \tilde{\xi}_t)$ *with initial state value* $(g_0, 1, 0)$.

C.1 *State variables* (M_t^*, ξ_t)

Assume that a sequential saddle-point that solves the policy problem exists.[31] Our objective is to transform the sequential saddle-point into a recursive saddle-point along the lines of Marcet and Marimon (2011). To achieve this, we augment the state space and modify properly the period return function associated with the sequential saddle-point.

Fix the multiplier on the implementability constraint (27) to a positive value, $\Phi > 0$, and form the partial Lagrangian

$$\tilde{L}_0 \equiv U(g_0) + \Phi\Omega_0(g_0) - \Phi U_c(g_0)b_0 + \beta\tilde{L},$$

where

$$\tilde{L} \equiv E_0 \sum_{t=1}^{\infty} \beta^{t-1}\big\{U_t + \Phi M_t^*\Omega_t - \xi_t[V_t - U_t - \beta(E_t m_{t+1}^* V_{t+1} + \theta E_t m_{t+1}^* \ln m_{t+1}^*)]\big\}.$$

Note that we do not include in the partial Lagrangian the law of motion of the likelihood ratio M_t^* (which is the reason why we distinguish in notation between $\tilde{L}_0$ in this section and L in Section B.2) and that we have already expressed labor in terms of consumption $h_t = c_t + g_t$ in $\tilde{L}_0$. Furthermore, we differentiate between the initial period and the rest of the periods due to the presence of initial debt and the realization of uncertainty at $t = 0$.

Bear in mind that we do not substitute for the optimal value of the conditional likelihood ratio m_t^* (20) in the household's utility recursion, which retains *linearity* with respect to the approximating model π in $\tilde{L}$. This allows us to apply the law of iterated expectations and rewrite $\tilde{L}$ in terms of current and lagged values of ξ_t:

$$\tilde{L} = E_0 \sum_{t=1}^{\infty} \beta^{t-1}[U_t + \Phi M_t^*\Omega_t - \xi_t(V_t - U_t) + \xi_{t-1}(m_t^* V_t + \theta m_t^* \ln m_t^*)]. \tag{45}$$

Consider the following saddle-point problem from period one onward.

Problem 1. We have

$$\min_{\xi_t, t\geq 1}\ \max_{c_t, m_t^*, M_t^*, V_t, t\geq 1} \tilde{L}$$

subject to

$$M_t^*(g^t) = m_t^*(g^t) M_{t-1}^*(g^{t-1}), \quad t \geq 1$$

$$m_t^*(g^t) = \frac{\exp(-V_t(g^t)/\theta)}{E_{t-1}\exp(-V_t(g^t)/\theta)}, \quad t \geq 1,$$

with initial values $M_0 = 1$, $\xi_0 = 0$, and g_0 given.

[31] The existence of a sequential saddle-point is not guaranteed due to the nonconvexity of the government's problem. However, if it exists, it solves the policy problem. See Marcet and Marimon (2011).

The modified return function in (45) does not depend on expectations of future variables, but only on the controls (c_t, m_t^*, V_t, ξ_t) and the lagged values (M_{t-1}^*, ξ_{t-1}), which serve as state variables. The object of interest is the value function of Problem 1, which is a solution to a saddle-point functional equation.

More precisely, assume that the approximating model of government expenditures is Markov with transition probabilities $\pi_{g|g_-} \equiv \text{Prob}(g_t = g | g_{t-1} = g_-)$ and let the vector $X_t \equiv (g_t, M_t^*, \xi_t)$ denote the state. Let $W(X_-; \Phi)$ denote the corresponding value function of the saddle-point problem when the state is X_-, where the underscore (_) stands for previous period, i.e., $z_- \equiv z_{t-1}$ for any random variable z. The value of Problem 1 is $W(g_0, 1, 0; \Phi)$; $\Phi > 0$ is treated as a parameter in the value function.

Bellman equation I. The function $W(\cdot; \Phi)$ satisfies the Bellman equation

$$
\begin{aligned}
&W(g_-, M_-^*, \xi_-; \Phi) \\
&\quad = \min_{\xi_g} \max_{c_g, m_g^*, V_g} \sum_g \pi_{g|g_-} \big\{ U(c_g, 1 - c_g - g) + \Phi m_g^* M_-^* \Omega_g \\
&\qquad - \xi_g (V_g - U(c_g, 1 - c_g - g)) + \xi_- (m_g^* V_g + \theta m_g^* \ln m_g^*) + \beta W(g, m_g^* M_-^*, \xi_g; \Phi) \big\},
\end{aligned}
$$

where

$$
\Omega_g \equiv [U_c(c_g, 1 - c_g - g) - U_l(c_g, 1 - c_g - g)] c_g - U_l(c_g, 1 - c_g - g) g
$$

and

$$
m_g^* = \frac{\exp(-V_g/\theta)}{\sum_g \pi_{g|g_-} \exp(-V_g/\theta)}, \quad \forall g.
$$

Time zero problem. The planner's problem at time zero takes the form

$$
\begin{aligned}
W_0(g_0, b_0; \Phi) = \max_{c_0} \{ &U(c_0, 1 - c_0 - g_0) + \Phi \Omega_0(c_0) \\
&- \Phi U_c(c_0, 1 - c_0 - g_0) b_0 + \beta W(g_0, 1, 0; \Phi) \},
\end{aligned}
$$

which is effectively the *static* problem

$$
\max_{c_0} U(c_0, 1 - c_0 - g_0) + \Phi \Omega_0(c_0) - \Phi U_c(c_0, 1 - c_0 - g_0) b_0.
$$

From the problem above, we get the initial period consumption, $c_0(g_0, b_0; \Phi)$.

Envelope conditions. The envelope conditions are

$$
W_{M^*}(g_-, M_-^*, \xi_-; \Phi) = \sum_g \pi_{g|g_-} m_g^* [\Phi \Omega_g + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi)] \tag{46}
$$

$$
W_\xi(g_-, M_-^*, \xi_-; \Phi) = \sum_g \pi_{g|g_-} [m_g^* V_g + \theta m_g^* \ln m_g^*]. \tag{47}
$$

Condition (47) exposes the connection between the shadow value ξ of manipulating the worst-case model and the promised utility to the household. Furthermore, solving (46) forward and converting to sequence notation allows us to conclude that

$$\begin{aligned} W_{M^*}(g_{t-1}, M^*_{t-1}, \xi_{t-1}; \Phi) &= \Phi E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M^*_{t+i}}{M^*_{t-1}} \Omega_{t+i} \\ &= \Phi E_{t-1} m^*_t \left[E_t \sum_{i=0}^{\infty} \beta^i \frac{M^*_{t+i}}{M^*_t} \Omega_{t+i} \right] \\ &= \Phi E_{t-1} m^*_t U_{ct} b_t, \end{aligned} \tag{48}$$

where, in the last line, we recognize the relationship between the present value of government surpluses and debt.

First-order conditions. For completeness, we derive the first-order conditions of the functional equation to verify that they match the first-order conditions of the sequential Lagrangian formulation. Assign the multiplier $\pi_{g|g_-}\tilde{\mu}_g$ on the optimal distortion m^*_g and get the first-order conditions

$$\begin{aligned} c_g: \quad & (U_{l,g} - U_{c,g})(1 + \xi_g + \Phi m^*_g M^*_-) \\ & = \Phi m^*_g M^*_- [(U_{cc} - 2U_{cl,g} + U_{ll,g})c_g + (U_{ll,g} - U_{cl,g})g] \end{aligned} \tag{49}$$

$$m^*_g: \quad \tilde{\mu}_g = \Phi M^*_- [\Omega_g + \beta W_{M^*}(g, M^*_g, \xi_g; \Phi)] + \xi_- [V_g + \theta(1 + \ln m^*_g)] \tag{50}$$

$$V_g: \quad \xi_g = \sigma m^*_g \left[\tilde{\mu}_g - \sum_g \pi_{g|g_-} m^*_g \tilde{\mu}_g \right] + m^*_g \xi_- \tag{51}$$

$$\xi_g: \quad V_g = U_g + \beta W_\xi(g, M^*_g, \xi_g; \Phi). \tag{52}$$

Equation (49) represents the familiar optimal wedge, with $h_g = c_g + g$. Furthermore, using the envelope condition with respect to ξ ((47)) in optimality condition (52) delivers the household's utility recursion (30). It remains to show that (51) describes the appropriate law of motion of the multiplier ξ_t. For that, consider at first (50) in sequence notation and use the fact that $\ln m^*_t = -V_t/\theta - \ln E_{t-1} \exp(-V_t/\theta)$ to get

$$\tilde{\mu}_t = M^*_{t-1}[\Phi\Omega_t + \beta W_{M^*}(g, M^*_t, \xi_t; \Phi)] + \xi_{t-1}\theta\left[1 - \ln E_{t-1} \exp\left(-\frac{V_t}{\theta}\right)\right].$$

Using (48), we see that $\Phi\Omega_t + \beta W_{M^*}(g_t, M^*_t, \xi_t; \Phi) = \Phi(\Omega_t + \beta E_t m^*_{t+1} U_{c,t+1} b_{t+1}) = \Phi U_{ct} b_t$. Thus

$$\tilde{\mu}_t = M^*_{t-1} \Phi U_{c_t} b_t + \xi_{t-1}\theta\left[1 - \ln E_{t-1} \exp\left(-\frac{V_t}{\theta}\right)\right],$$

with innovation

$$\tilde{\mu}_t - E_{t-1} m^*_t \tilde{\mu}_t = M^*_{t-1} \Phi (U_{ct} b_t - E_{t-1} m^*_t U_{ct} b_t),$$

since the term multiplying ξ_{t-1} is known with respect to information at $t-1$. Plugging the innovation of $\tilde{\mu}$ into (51) delivers the law of motion (38).

Policy functions and debt. Given the recursive representation of the government's problem, we attain a time invariant representation of the policy functions as functions of the state, e.g., the optimal policy function for consumption is $c_g = c_g(g_-, M_-^*, \xi_-; \Phi)$. In the case of an independent and identically distributed approximating model, we can drop the dependence on g_-. Note though that (49) shows that (g, M_g^*, ξ_g) is sufficient to determine c. Thus, the vector of state variables (g_-, M_-^*, ξ_-) affects the optimal policy for consumption at g by determining the value of the *current* state (g, M_g^*, ξ_g) and, consequently, $c_g = c_g(g_-, M_-^*, \xi_-; \Phi) = c(g, M_g^*, \xi_g; \Phi)$. Therefore, labor and the optimal tax rate also depend on the current values of the state. Note also that (52) allows us to use the same logic with the household's utility, so $V_g = V(g, M_g^*, \xi_g; \Phi)$. Turning to debt, using (48) allows us to determine the optimal debt position as a function of the current state $b_t = b(g_t, M_t^*, \xi_t; \Phi)$, since

$$b_t = \frac{\Omega_t}{U_{ct}} + \frac{\beta}{\Phi U_{ct}} W_{M^*}(g_t, M_t^*, \xi_t; \Phi).$$

To conclude, remember that the recursive formulation has been contingent on the value $\Phi > 0$. After the initial period problem and the functional problem are solved, Φ has to be adjusted so that the intertemporal budget constraint is satisfied. The expression that we derived for optimal debt suggests the use of the derivative W_{M^*} for that purpose: Increase (decrease) Φ if $\Omega_0/U_{c0} + (\beta/(\Phi U_{c0}))W_{M^*}(g_0, 1, 0; \Phi) - b_0 < \ (>)\ 0$. This procedure has to be repeated, and the initial period problem and the functional equation have to be resolved until the intertemporal budget constraint holds with equality.

C.2 *Normalized multiplier* $\tilde{\xi}_t$

The same methodology allows us to derive a recursive representation in terms of the normalized multiplier $\tilde{\xi}_t$. Form the partial Lagrangian by multiplying the household's utility recursion (30) with M_t^* and assign to this constraint the multiplier $\beta^t \pi_t \tilde{\xi}_t$, with $\tilde{\xi}_0 \equiv 0$. Now follow similar steps as in the previous subsection to get the functional equation:

Bellman equation II. We have

$$\begin{aligned} J(g_-, M_-^*, \tilde{\xi}_-; \Phi) = \min_{\tilde{\xi}_g} \max_{c_g, m_g^*, V_g} \sum_g \pi_{g|g_-} \big[& U(c_g, 1 - c_g - g) + \Phi m_g^* M_-^* \Omega_g \\ & - m_g^* M_-^* \tilde{\xi}_g (V_g - U(c_g, 1 - c_g - g)) \\ & + \tilde{\xi}_- M_-^* (m_g^* V_g + \theta m_g^* \ln m_g^*) + \beta J(g, m_g^* M_-^*, \tilde{\xi}_g; \Phi) \big], \end{aligned}$$

where Ω_g and m_g^* as before.

Envelope conditions. We have

$$\begin{aligned} J_{M^*}(g_-, M_-^*, \tilde{\xi}_-; \Phi) = \sum_g \pi_{g|g_-} [& \Phi m_g^* \Omega_g - m_g^* \tilde{\xi}_g (V_g - U_g) + \tilde{\xi}_- (m_g^* V_g + \theta m_g^* \ln m_g^*) \\ & + \beta m_g^* J_{M^*}(g, M_g^*, \tilde{\xi}_g; \Phi)] \end{aligned} \tag{53}$$

$$J_{\tilde{\xi}}(g_-, M_-^*, \tilde{\xi}_-; \Phi) = M_-^* \sum_g \pi_{g|g_-} (m_g^* V_g + \theta m_g^* \ln m_g^*). \tag{54}$$

Matching first-order conditions. Assign multiplier $\pi_{g|g_-}\hat{\mu}_g$ on the conditional distortion of the household m^*_g and derive the first-order conditions

$$c_g:\quad (U_{l,g}-U_{c,g})(1/M^*_g+\tilde{\xi}_g+\Phi) = \Phi[(U_{cc,g}-2U_{cl,g}+U_{ll,g})c_g+(U_{ll,g}-U_{cl,g})g] \tag{55}$$

$$m^*_g:\quad \hat{\mu}_g = M^*_-\big[\Phi\Omega_g-\tilde{\xi}_g(V_g-U_g)+\tilde{\xi}_-(V_g+\theta(\ln m^*_g+1))+\beta J^*_M(g,M^*_g,\tilde{\xi}_g;\Phi)\big] \tag{56}$$

$$V_g:\quad \tilde{\xi}_g M^*_- = \sigma\bigg(\hat{\mu}_g-\sum_g \pi_{g|g_-}m^*_g\hat{\mu}_g\bigg)+\tilde{\xi}_-M^*_- \tag{57}$$

$$\tilde{\xi}_g:\quad m^*_gM^*_-V_g = m^*_gM^*_-U_g+\beta J_{\tilde{\xi}}(g,M^*_g,\tilde{\xi}_g;\Phi). \tag{58}$$

Condition (55) describes the familiar optimal wedge. Now turn to sequence notation, update the envelope condition (54) one period, substitute in (58), and simplify to get the household's utility recursion

$$V_t = U_t+\beta(E_tm^*_{t+1}V_{t+1}+\theta E_tm^*_{t+1}\ln m^*_{t+1}).$$

There is some work needed in order to derive the law of motion of the multiplier $\tilde{\xi}_t$ in the text. Consider the envelope condition (53) and solve it forward to get

$$\begin{aligned} J_{M^*}(g_{t-1},M^*_{t-1},\tilde{\xi}_{t-1};\Phi) &= \Phi E_{t-1}\sum_{i=0}^{\infty}\beta^i\frac{M^*_{t+i}}{M^*_{t-1}}\Omega_{t+i} \\ &\quad - E_{t-1}\sum_{i=0}^{\infty}\beta^i\frac{M^*_{t+i}}{M^*_{t-1}}\tilde{\xi}_{t+i}(V_{t+i}-U_{t+i}) \\ &\quad + E_{t-1}\sum_{i=0}^{\infty}\beta^i\frac{M^*_{t+i-1}}{M^*_{t-1}}\tilde{\xi}_{t+i-1}(m^*_{t+i}V_{t+i}+\theta m^*_{t+i}\ln m^*_{t+i}). \end{aligned}$$

The last sum in the third line can be rewritten as

$$\begin{aligned} &E_{t-1}\sum_{i=0}^{\infty}\beta^i\frac{M^*_{t+i-1}}{M^*_{t-1}}\tilde{\xi}_{t+i-1}(m^*_{t+i}V_{t+i}+\theta m^*_{t+i}\ln m^*_{t+i}) \\ &\quad = \tilde{\xi}_{t-1}E_{t-1}(m^*_tV_t+\theta m^*_t\ln m^*_t) \\ &\qquad + E_{t-1}\sum_{i=0}^{\infty}\beta^i\tilde{\xi}_{t+i}\beta(m^*_{t+i+1}V_{t+i+1}+\theta m^*_{t+i+1}\ln m^*_{t+i+1}). \end{aligned}$$

Thus the derivative of the value function with respect to the likelihood ratio M^* becomes

$$\begin{aligned} &J_{M^*}(g_{t-1},M^*_{t-1},\tilde{\xi}_{t-1};\Phi) \\ &\quad = \Phi E_{t-1}\sum_{i=0}^{\infty}\beta^i\frac{M^*_{t+i}}{M^*_{t-1}}\Omega_{t+i}+\tilde{\xi}_{t-1}E_{t-1}(m^*_tV_t+\theta m^*_t\ln m^*_t) \end{aligned}$$

$$- E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M^*_{t+i}}{M^*_{t-1}} \tilde{\xi}_{t+i}(V_{t+i} - U_{t+i} - \beta E_{t+i}(m^*_{t+i+1} V_{t+i+1} + \theta m^*_{t+i+1} \ln m^*_{t+i+1}))$$
$$= \Phi E_{t-1} m^*_t U_{ct} b_t + \tilde{\xi}_{t-1} E_{t-1}(m^*_t V_t + \theta m^*_t \ln m^*_t),$$

by using the household's utility recursion, and the relationship between debt and the present value of future government surpluses.

Update J_{M^*} one period and plug it into the first-order condition (56) to get

$$\begin{aligned} \hat{\mu}_t = M^*_{t-1}\big[&\Phi(\Omega_t + \beta E_t m^*_{t+1} U_{c,t+1} b_{t+1}) \\ &- \tilde{\xi}_t (V_t - U_t - \beta(E_t m^*_{t+1} V_{t+1} + \theta E_t m^*_{t+1} \ln m^*_{t+1})) + \tilde{\xi}_{t-1}(V_t + \theta(\ln m^*_t + 1))\big] \\ = M^*_{t-1}\big[&\Phi(\Omega_t + \beta E_t m^*_{t+1} U_{c,t+1} b_{t+1}) + \tilde{\xi}_{t-1}(V_t + \theta(\ln m^*_t + 1))\big], \end{aligned}$$

again using the household's utility recursion. Note that $\Omega_t + \beta E_t m^*_{t+1} U_{c,t+1} b_{t+1} = U_{ct} b_t$. Now use the expression for the conditional distortion m^*_t to finally get

$$\hat{\mu}_t = M^*_{t-1}\big[\Phi U_{ct} b_t + \tilde{\xi}_{t-1} \theta (1 - \ln E_{t-1} \exp(\sigma V_t))\big].$$

Therefore, the innovation in $\hat{\mu}_t$ becomes $\hat{\mu}_t - E_{t-1} m^*_t \hat{\mu}_t = \Phi M^*_{t-1}[U_{ct} b_t - E_{t-1} m^*_t U_{ct} b_t]$. Plugging the innovation into (57) and simplifying delivers the law of motion of the normalized multiplier (39).

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