Calculations for Theorem 2 in 'Competing auctions ...' for Theoretical Economics 5 (2010)

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Let us rewrite the key formula from the Appendix of the paper:

$$\gamma'(a) = -(n-1)(1-\frac{1}{k})^{n-2}\frac{1}{k} - \frac{(n-1)(n-2)}{k^2}af(a)(1-\frac{1}{k})^{n-3}+$$

$$+\frac{(n-1)}{k}af(a)(1-\frac{1}{k})^{n-2} + \frac{n-1}{k(k-1)}r^*f(a)(1-\frac{1}{k})^{n-2}+$$

$$+\frac{n-1}{k(k-1)}(1-\frac{1}{k})^{n-2}r^*f(a) + \frac{(n-1)(n-2)}{k^2(k-1)^2}(1-\frac{1}{k})^{n-3}r^*f(a)+$$

$$+\frac{n-1}{k-1}(1-\frac{1}{k})^{n-2} - \frac{n-1}{k-1}af(a)(1-\frac{1}{k})^{n-2} + \frac{(n-1)(n-2)}{k(k-1)}af(a)(1-\frac{1}{k})^{n-3}-$$

$$-\frac{n-1}{k-1}af(a)(1-\frac{1}{k})^{n-2} - \frac{n-1}{k-1}af(a)\frac{n-2}{k(k-1)}(1-\frac{1}{k})^{n-3}.$$

As we argued in that Appendix, we need to find a condition for $\gamma'(a) \leq 0$ to hold when one substitutes $r^* = \frac{a(n-1)}{(n-1)+(k-1)^2}$. We start with the observation that if one divides $\gamma'(a)$ through by $(n-1)\frac{1}{k}(1-\frac{1}{k})^{n-3}$, then the sign of the expression does not change, and thus we can look for conditions under which $\frac{\gamma'(a)}{(n-1)\frac{1}{k}(1-\frac{1}{k})^{n-3}} \leq 0$. After substituting $r^* = \frac{a(n-1)}{(n-1)+(k-1)^2}$, we can write that

$$\frac{\gamma'(a)}{(n-1)\frac{1}{k}(1-\frac{1}{k})^{n-3}} = -(1-\frac{1}{k}) - \frac{n-2}{k}af(a) + af(a)(1-\frac{1}{k})+$$

$$+\frac{(n-1)}{(n-1)+(k-1)^2}\frac{1}{k-1}(1-\frac{1}{k})af(a) + \frac{1-\frac{1}{k}}{k-1}\frac{(n-1)}{(n-1)+(k-1)^2}af(a)+$$

$$+\frac{n-2}{k(k-1)^2}\frac{(n-1)}{(n-1)+(k-1)^2}af(a) + \frac{k}{k-1}(1-\frac{1}{k})-$$

$$-\frac{k}{k-1}af(a)(1-\frac{1}{k}) + \frac{n-2}{k-1}af(a) - \frac{k}{k-1}af(a)(1-\frac{1}{k}) - \frac{n-2}{(k-1)^2}af(a).$$

After collecting terms this expression becomes

$$\frac{\gamma'(a)}{(n-1)\frac{1}{k}(1-\frac{1}{k})^{n-3}} = \frac{1}{k}+$$

$$+af(a)[-\frac{n-2}{k}+1-\frac{1}{k}+\frac{2}{k}\frac{(n-1)}{(n-1)+(k-1)^2}+$$

$$+\frac{n-2}{k(k-1)^2}\frac{(n-1)}{(n-1)+(k-1)^2}-1+\frac{n-2}{k-1}-1-\frac{n-2}{(k-1)^2}] =$$

$$= \frac{1}{k}+af(a)[-1-\frac{1}{k}-\frac{n-2}{k(k-1)^2}+$$

$$+\frac{n-2}{k(k-1)^2}\frac{(n-1)}{(n-1)+(k-1)^2}+\frac{2}{k}\frac{(n-1)}{(n-1)+(k-1)^2}] =$$

$$= \frac{1}{k}+af(a)[-1-\frac{1}{k}+\frac{\frac{2(n-1)}{k}-\frac{n-2}{k}}{(n-1)+(k-1)^2}] =$$

$$= \frac{1}{k}+af(a)[-1-\frac{1}{k}+\frac{n/k}{(n-1)+(k-1)^2}].$$

This expression to be nonpositive is equvalent to

$$af(a)[k+1-\frac{n}{(n-1)+(k-1)^2}] \geq 1$$

or

$$af(a) \geq \frac{1}{k+1-\frac{n}{(n-1)+(k-1)^2}} =$$

$$= \frac{(n-1)+(k-1)^2}{(k+1)(n-1)+(k+1)(k-1)^2-n} =$$

$$= \frac{k(k-1)+n-k}{kn+n-k-1+k^3-k^2-k+1-n} =$$

$$= \frac{k(k-1)+n-k}{k^3-k^2+k(n-2)} = \frac{k-1+\frac{n-k}{k}}{k^2-2+n-k},$$

which concludes the proof.