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Endogenous indeterminacy and volatility of asset prices under ambiguity

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If agents are ambiguity-averse and can invest in productive assets, asset prices can robustly exhibit indeterminacy in the markets that open after the productive investment has been launched. For indeterminacy to occur, the aggregate supply of goods must appear in precise configurations, but the investment levels that generate these supplies arise systematically. That indeterminacy arises only at a knife-edge set of aggregate supplies allows for a simple explanation of the volatility of asset prices: small changes in supplies necessarily lead to a large price response.

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JEL classification. D51, D53, D81, G12.

1. Introduction

In a seminal paper, Dow and Werlang (1992) argue that ambiguity aversion can lead asset prices to be indeterminate. Ambiguity-averse agents deem a set of probability distributions to be possible and in the maximin formulation evaluate their asset portfolios using the worst-case distribution that minimizes their expected utility. With two states—an asset with either a high return or a low return—an agent whose initial endowment is state-invariant evaluates an asset purchase using the distribution that assigns the highest probability to the low-return state and evaluates an asset sale using the distribution that assigns the highest probability to the high-return state. This switch of distributions as agents contemplate going long or short can lead agents to stay out of the market over a range of prices ("portfolio inertia"), with the result that asset prices are indeterminate. Epstein and Wang (1994) pursue a related line of argument.

These conclusions face a difficulty however: for most consumption bundles, agents identify a single probability distribution as the worst case and this distribution remains the worst case for any small change in consumption. Locally, therefore, some or all of the agents in an ambiguity-averse society have smooth indifference curves and act just like classical expected-utility maximizers. These agents, moreover, determine the market response to price changes. For example, with one good in each of two states, an agent

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whose consumption varies by state has just one worst-case distribution locally. As long as the aggregate endowment varies by state, there must be at least one such agent in any competitive equilibrium, and that is enough to ensure that equilibria are typically determinate. The case for indeterminacy therefore seems easy to dismiss. These arguments appear in different ways in Epstein and Wang (1994), Chateauneuf et al. (2000), Dana (2004), and Rigotti and Shannon (2012), although they are not always directed to the Dow–Werlang no-trade argument. Dana and Rigotti–Shannon argue specifically that ambiguity-averse economies typically have determinate equilibria, just like economies of classical expected-utility maximizers.

We take another look at asset pricing under ambiguity aversion by departing from the standard pure-exchange setting in which endowments are exogenously given. Instead, agents have the option to invest in productive assets that endogenously alter the economy's state-by-state endowment of goods. It is then a robust event for equilibrium investment to occur at just the unusual levels that lead to asset-price indeterminacy: in the two-state case, the investment level that makes society's aggregate supply of goods constant across states. Asset prices then are indeterminate, not in the overall intertemporal equilibrium, but in the markets that open later, after the productive investment has been launched.[1] And the indeterminacy is real: rather than displaying portfolio inertia, agents trade in equilibrium and consequently variations in equilibrium asset prices change each agent's demand, consumption, and utility. The indeterminacy that arises endogenously lies in the class considered in Dana (2004) even though, as Dana pointed out, the indeterminacy is nongeneric in an exchange setting.

Investment robustly occurs in the special configurations that lead to indeterminacy due to the very fact that agents exhibit ambiguity aversion. In the two-state case, as an agent's consumption rises in the bad state where it is initially low, the agent switches the probability distribution he or she uses to evaluate asset portfolios at the point where consumption becomes equal across states. Consequently the utility return to an investment that enhances output at the bad state falls discontinuously at exactly the point where investment equalizes aggregate consumption across states. This discrete fall makes the special consumption-equalizing level of investment a systematic occurrence. When there are more than two states, the same scenario can unfold where investment occurs at just the level that leads to indeterminacy, but now agents' consumption need not be perfectly hedged and can vary by state.

The knife-edge feature of the indeterminacy in this paper—that indeterminacy occurs only at particular investment levels—heightens its economic relevance: arbitrarily small changes in quantities necessarily have a big impact on prices. Suppose we add a small amount of noise to the state-specific output of the investment technology. Although for almost every outcome of the noise, indeterminacy is then absent, asset prices will be volatile instead: if agents learn that there will be a small increase in output at some state, then—because of the switch in the probability distribution used to evaluate consumption portfolios—the price of assets with payoffs that are weighted toward

[1] This type of indeterminacy can occur in the absence of uncertainty when technology is modeled using production activities (Mandler 1995).

that state will fall discretely. The investments that ambiguity-averse agents undertake to hedge against uncertainty thus end up magnifying the uncertainty of prices. Indeed, no matter how small the production noise is, asset prices will display nonvanishing variance. The volatility of asset prices is, therefore, much larger than can be explained by the volatility of fundamentals: there is "excess volatility." In an economy of smooth expected-utility maximizers, in contrast, a small amount of noise leads to only a small variance in asset prices. We can therefore draw a bright line between the market equilibrium consequences of classical and ambiguity-averse agents.

The large price impact of small changes in quantities also means that agents have a strong incentive to manipulate market prices. In the working-paper version of this article, we illustrate this point by showing that no matter how small an agent is as a fraction of the entire economy, he or she can achieve a discrete utility gain by disposing of an arbitrarily small quantity of his or her endowment. A large economy of ambiguity-averse agents therefore cannot function competitively.

The distinctive character of the knife-edge indeterminacy that occurs with ambiguity aversion can be seen in a graph of the equilibrium correspondence (the map from exogenous parameters to equilibrium prices and quantities). For a two-state economy of agents with maximin ambiguity aversion, Figure 1 pictures the map from some agent i's endowment of the good that appears at a state b to that good's normalized equilibrium price. Assume, when i's endowment equals $\overline{e}^i_b$ and fixing the endowments of the other agents, that the aggregate endowment of the state b good equals the aggregate endowment of the economy's other state-contingent good. Indeterminacy is then present at $\overline{e}^i_b$ while at nearby endowments, equilibrium prices are unique. Although the endowment that leads to indeterminacy is rare, we will see that it arises systematically. Just as importantly, small variations in e^i_b in the neighborhood of $\overline{e}^i_b$ necessarily lead to large changes in equilibrium prices; this feature of the model drives our volatility results. A discontinuity of prices is unavoidable due to the fact that the equilibrium correspondence does not admit a continuous selection in a neighborhood of $\overline{e}^i_b$, which, in turn, is due to the failure of the equilibrium correspondence to be lower hemicontinuous at $\overline{e}^i_b$. If, in contrast, the equilibrium correspondence were continuous, then it would admit a continuous selection and hence prices could be a stable function of endowments (Figure 2). That the link between indeterminacy and volatility is mandatory in Figure 1 but not in Figure 2 is one of this paper's main points. We illustrate this contrast via the Bewley (2002) incomplete-preferences model of Knightian uncertainty, where indeterminacy is omnipresent but the equilibrium correspondence is continuous.

To sum up, it has long been clear that the Dow–Werlang no-trade argument does not by itself generate equilibrium indeterminacy or volatility; "some other ingredient has to be inserted" in the words of Mukerji and Tallon (2004). One way to fill the gap is to let assets have an idiosyncratic component to their return that agents regard as ambiguous, as in Epstein and Wang (1994) and Mukerji and Tallon (2001). Another way to use ambiguity to explain volatility is to consider the effect on equilibrium prices of a parameter such as a signal that can take on infinitely many values (see Epstein and Schneider 2008, Illeditsch 2011, Condie and Ganguli 2011). With this second path, however, ambiguity

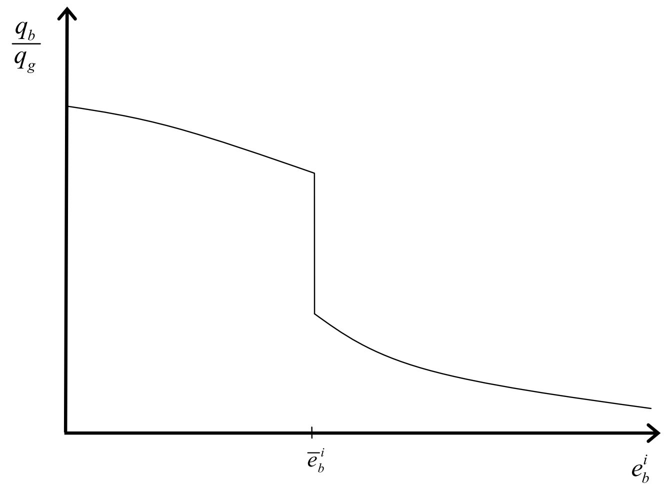

FIGURE 1. Endogenous indeterminacy (failure of lower hemicontinuity).

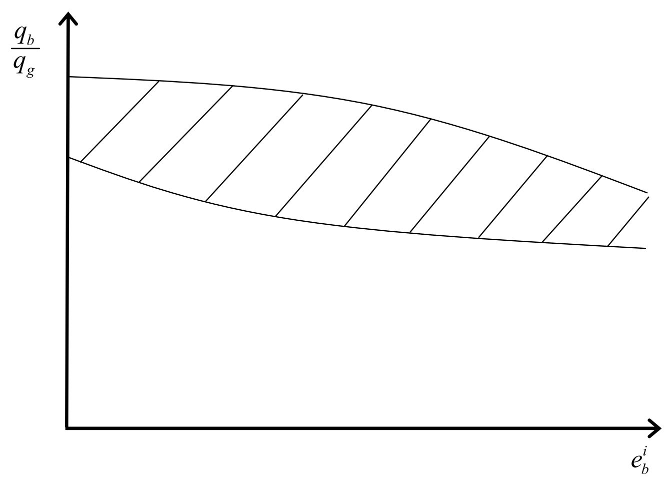

FIGURE 2. Indeterminacy with incomplete preferences (full continuity).

aversion no longer plays a distinctive role. In a traditional general-equilibrium model, if a parameter sweeps through infinitely many values, a point can come where a small parameter change has to induce a large change in prices. Consider Figure 1 again or even the more orthodox case where the equilibrium correspondence is S-shaped. In a family of infinitely many economies, therefore, it is not surprising to find that at some point a small parameter change brings a large price response. The aim of this paper, in contrast, is to show that indeterminacy and volatility can arise robustly in a neighborhood of a *single* model. The extra ingredient needed to deliver this goal is nothing more than the presence of a productive asset.

2. A simple economy

Although indeterminacy and volatility arise robustly in economies with ambiguity-averse agents, there are certainly some combinations of agents and technology parameters that lead to well behaved equilibria where, in both the overall intertemporal economy and in the economy's later periods of operation, prices and allocations are locally unique and change smoothly as a function of output levels. Since indeterminacy is not always present, there is no loss in turning to the simplest framework that is rich enough for asset price indeterminacy and volatility to be robust.

In period 1, agents can use their resources either to consume or to invest in a productive asset that generates output in the final period, period 2. Between periods 1 and 2 is an intermediate period in which agents can trade state-contingent claims on period 2 goods. We could let there be further consumption in the intermediate period without changing any result.

There are two states g and b (for good and bad) and uncertainty is resolved between the intermediate period and period 2. So altogether there are three consumption goods: x_1, consumed in period 1; x_b, consumed in period 2 in state b; and x_g, consumed in period 2 in state g. Generic consumption in period 2 is labeled x_2.

There is a finite set of agents $\mathcal{I}$. Agent $i \in \mathcal{I}$ is endowed with the quantities $(e^i_1, e^i_b, e^i_g) \gg 0$ of the three goods where $\sum_{i\in\mathcal{I}} e^i_g > \sum_{i\in\mathcal{I}} e^i_b$, which is why b is called the bad state. The preferences of agent i are described by a differentiably strictly concave and strictly increasing function $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ that gives the utility index of goods in the two time periods[2] and a closed interval of probabilities $\mathcal{P}^i = [\underline{\pi}^i, \overline{\pi}^i]$ for state b, where $0 < \underline{\pi}^i < \overline{\pi}^i < 1$. Ambiguity aversion takes the form of a Gilboa and Schmeidler (1989) maximin assumption on preferences over state-contingent bundles: each i has the utility function $U^i : \mathbb{R}^3_+ \to \mathbb{R}$ defined by

$$U^i(x^i_1, x^i_b, x^i_g) = \min_{\pi\in\mathcal{P}^i} [\pi u^i(x^i_1, x^i_b) + (1-\pi) u^i(x^i_1, x^i_g)]. \tag{1}$$

The productive asset links the periods: if agent i invests k^i of the first-period good in the productive asset, then i receives a return of ck^i units of x_b, where $c > 0$. So the quantity of goods available in period 2 in state b is $\sum_{i\in\mathcal{I}}(ck^i + e^i_b)$. One may think of the productive asset as an investment that alleviates the bad consequences of state b. As we explain, the key feature of the asset is that it changes the mix of state b and g goods that occurs in the agents' natural endowments in period 2. It is only for simplicity that we assume that the asset yields output in just one state.

Since we are interested only in robust cases of asset-price indeterminacy and volatility, we need to distinguish persistent properties of models from flukes. A *model* is a $\omega = ((u^i, \underline{\pi}^i, \overline{\pi}^i, e^i_1, e^i_b, e^i_g)_{i\in\mathcal{I}}, c)$ that meets the assumptions we have stated and Ω is the set of models. We give Ω a topology by defining ω_n to converge to ω ($\omega_n \to \omega$) if and only if $((\underline{\pi}^i(n), \overline{\pi}^i(n), e^i_1(n), e^i_b(n), e^i_g(n))_{i\in\mathcal{I}}, c(n)) \to ((\underline{\pi}^i, \overline{\pi}^i, e^i_1, e^i_b, e^i_g)_{i\in\mathcal{I}}, c)$ in the Euclidean sense and, for each u^i and compact $K \subset \mathbb{R}^2_+$, $(u^i(n), Du^i(n))$ converges uniformly to (u^i, Du^i) on K. A property of a model ω is *robust* if there is an open neighborhood Ω' of ω such that the property holds for all models in Ω'.

[2]Differentiable strict concavity means that $D^2u^i(x^i)$ is negative definite for all $x^i \in \mathbb{R}^2_+$.

3. Equilibrium and indeterminacy

In a market equilibrium of the model in Section 2, the agents in period 1 can use their first-period endowment to buy shares in a firm that invests the first-period good in production. Each share in the firm is used to buy one unit of the first-period good and so shares in the first period have the same price as the first-period good. Let θ^i_1 be agent i's share purchase, which can be negative. In the intermediate period, the agent can trade shares in the firm and assets that deliver output in period 2: one asset that delivers a unit of output in state b and another that delivers a unit of output in state g. Agent i's purchases of these assets are given by θ^i_b and θ^i_g and their prices are labeled p_b and p_g. Since a share in the firm delivers c units of the state b good in period 2, it must sell for $p_b c$ in the intermediate period. To keep the accounting simple, we do not distinguish in the intermediate period between a single unit of the asset that delivers the state b good and c units of firm shares. So the aggregate purchases in the intermediate period of the state b asset must sum to the aggregate supply of firm shares multiplied by c.

With this market structure, agent i faces the budget constraint $x^i_1+\theta^i_1\leq e^i_1$ in the first period, $p_b\theta^i_b+p_g\theta^i_g\leq p_b c\theta^i_1$ in the intermediate period, and $x^i_b\leq e^i_b+\theta^i_b$ and $x^i_g\leq e^i_g+\theta^i_g$ in states b and g in the second period. So, given $(p_b, p_g, e^i_1, e^i_b, e^i_g)$, agent i's budget set is given by

$$B^i=\big\{(x^i_1, x^i_b, x^i_g, \theta^i_1, \theta^i_b, \theta^i_g)\in\mathbb{R}^3_+\times\mathbb{R}^3: \\ x^i_1+\theta^i_1\leq e^i_1,\ p_b\theta^i_b+p_g\theta^i_g\leq p_b c\theta^i_1,\ x^i_b\leq e^i_b+\theta^i_b,\ x^i_g\leq e^i_g+\theta^i_g\big\}.$$

Markets are complete in the standard sense that the span of the bundles that can be reached by trading the assets θ_b and θ_g equals all of $\mathbb{R}^2$.

Definition 1. An *equilibrium* is a $((x^i_1, x^i_b, x^i_g, \theta^i_1, \theta^i_b, \theta^i_g)_{i\in\mathcal{I}}, p_b\geq 0, p_g\geq 0)$, where $p_b+p_g=1$, such that:

- For each $i\in\mathcal{I}$, $(x^i_1, x^i_b, x^i_g, \theta^i_1, \theta^i_b, \theta^i_g)\in B^i$ and $U^i(x^i_1, x^i_b, x^i_g)\geq U^i(x^{i\prime}_1, x^{i\prime}_b, x^{i\prime}_g)$ for all $(x^{i\prime}_1, x^{i\prime}_b, x^{i\prime}_g, \theta^i_1, \theta^{i\prime}_b, \theta^{i\prime}_g)\in B^i$,
- $\sum_{i\in\mathcal{I}}(x^i_1+\theta^i_1)=\sum_{i\in\mathcal{I}}e^i_1$, $\sum_{i\in\mathcal{I}}\theta^i_1\geq 0$, $\sum_{i\in\mathcal{I}}\theta^i_b=\sum_{i\in\mathcal{I}}c\theta^i_1$, $\sum_{i\in\mathcal{I}}\theta^i_g=0$, $\sum_{i\in\mathcal{I}}x^i_b=\sum_{i\in\mathcal{I}}(e^i_b+\theta^i_b)$, $\sum_{i\in\mathcal{I}}x^i_g=\sum_{i\in\mathcal{I}}(e^i_g+\theta^i_g)$.

Any model that satisfies our assumptions has an equilibrium.

In a standard perfect-foresight interpretation of equilibrium, agents know p_b and p_g in the initial period, even though the markets where those prices rule have not yet opened. But once we come to the intermediate period, will those initial-period expectations form determinate equilibrium prices? If not, then how can market forces by themselves lead those expectations to rule as market prices? Since agents arrive in the intermediate period with the standard characteristics of general-equilibrium consumers—endowments and preferences—we can ask whether equilibria for the intermediate-period markets are locally unique and, specifically, if there are prices in the intermediate period that clear markets that differ slightly from the prices that were anticipated in the initial period. While, in a classical general-equilibrium model, indeterminacy arises only at a measure 0 set of parameters—and thus is dismissible—the endowments of an intermediate economy are not arbitrary; they are endogenously determined by the agents' first-period equilibrium investment decisions.

When prices are not constrained to fulfill initial-period expectations, we denote them by (q_b, q_g). To determine the (q_b, q_g) that can clear markets in the intermediate period, fix some equilibrium $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$. Agent i in the intermediate period then chooses $(x_b^i, x_g^i, \theta_b^i, \theta_g^i) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ to maximize $U^i(\overline{x}_1^i, x_b^i, x_g^i)$ subject to

$$(x_b^i, x_g^i, \theta_b^i, \theta_g^i) \in B_{\text{int}}^i$$
$$\equiv \{(x_b^i, x_g^i, \theta_b^i, \theta_g^i) \in \mathbb{R}_+^2 \times \mathbb{R}^2 : q_b\theta_b^i + q_g\theta_g^i \le q_b c\overline{\theta}_1^i, x_b^i \le e_b^i + \theta_b^i, x_g^i \le e_g^i + \theta_g^i\}.$$

Definition 2. Given the equilibrium $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$, an *intermediate equilibrium* is $((x_b^i, x_g^i, \theta_b^i, \theta_g^i)_{i\in\mathcal{I}}, q_b \ge 0, q_g \ge 0)$, where $q_b + q_g = 1$ such that:

- For each $i \in \mathcal{I}$, $(x_b^i, x_g^i, \theta_b^i, \theta_g^i) \in B_{\text{int}}^i$ and $U^i(\overline{x}_1^i, x_b^i, x_g^i) \ge U^i(\overline{x}_1^i, x_b^{i\prime}, x_g^{i\prime})$ for all $(x_b^{i\prime}, x_g^{i\prime}, \theta_b^{i\prime}, \theta_g^{i\prime}) \in B_{\text{int}}^i$.
- $\sum_{i\in\mathcal{I}} \theta_b^i = \sum_{i\in\mathcal{I}} c\overline{\theta}_1^i$, $\sum_{i\in\mathcal{I}} \theta_g^i = 0$, $\sum_{i\in\mathcal{I}} x_b^i = \sum_{i\in\mathcal{I}}(e_b^i + \theta_b^i)$, $\sum_{i\in\mathcal{I}} x_g^i = \sum_{i\in\mathcal{I}}(e_g^i + \theta_g^i)$.

It is easy to confirm, given the equilibrium $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$, that $((\overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, q_b = \overline{p}_b, q_g = \overline{p}_g)$ is an intermediate equilibrium. When other intermediate equilibria exist arbitrarily near to $((\overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$, then there is "endogenous" indeterminacy in the intermediate period.

Definition 3. An equilibrium $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$ is *indeterminate along the equilibrium path* if there is a continuum of intermediate equilibria that contains $((\overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$.[3]

Since preferences are strictly convex, any pair of distinct intermediate equilibria must have different price vectors. Definition 3 does not impose a new equilibrium concept: Definition 1 remains in force. Indeterminacy along the equilibrium path is simply a property of a conventional equilibrium for intertemporal models, although not a property that has received much attention in general-equilibrium theory.

Given an intermediate equilibrium, the price of any asset beyond our simple contracts that deliver output in one state only is determined by q_b and q_g. For linear combinations of our simple contracts, indeterminacy along the equilibrium path leads to an indeterminacy of the intermediate-period price of almost any asset in this class.

[3]A set of intermediate equilibria $E \subset \mathbb{R}^{4I+2}$ forms a continuum if $|E| > 1$ and there is a continuous function f from some interval $T \subset \mathbb{R}$ onto E.

Proposition 1. *There are models where it is robust for all equilibria to be indeterminate along the equilibrium path.*

To grasp why Proposition 1 is true, suppose for the moment that it is robust for agents in equilibrium to invest just enough to make the aggregate supply of output in the b state equal the aggregate endowment in the g state,

$$\sum_{i\in\mathcal{I}}(c\overline{\theta}_1^i + e_b^i) = \sum_{i\in\mathcal{I}} e_g^i. \tag{2}$$

Consistent with (2), each agent i can consume a bundle with $\overline{x}_b^i = \overline{x}_g^i$ at the equilibrium price vector $(\overline{p}_b, \overline{p}_g)$. If, in that case, we view the economy that operates in the intermediate period as a model in its own right, its relative price ratio q_b/q_g is not pinned down by the intermediate period's market-clearing requirements. For any (x_b^i, x_g^i) with $x_b^i = x_g^i$, any $\pi \in \mathcal{P}^i$ solves the minimization problem in (1) and it follows that any such (x_b^i, x_g^i) is supported by a continuum of price vectors, specifically any price ratio in $[\underline{\pi}^i/(1-\underline{\pi}^i), \overline{\pi}^i/(1-\overline{\pi}^i)]$.[4] Agent i's indifference curve is therefore kinked at any state-invariant bundle. If $\overline{p}_b/\overline{p}_g$ lies in $\bigcap_{i\in\mathcal{I}}(\underline{\pi}^i/(1-\underline{\pi}^i), \overline{\pi}^i/(1-\overline{\pi}^i))$, then each agent in the intermediate period continues to demand a state-invariant bundle following a slight change in q_b/q_g away from $\overline{p}_b/\overline{p}_g$. The sum of the reduced-form budget constraints that hold in the intermediate period[5] and (2) then imply that the markets for both the state b and g goods continue to clear following the change in q_b/q_g.

But why should it be robust for (2) to be satisfied? As aggregate investment $\sum_{i\in\mathcal{I}}\theta_1^i$ passes through the level at which (2) holds, at least some agents must switch from using the distribution with the highest probability of state b to the distribution with the lowest probability of state b. The resulting discrete fall in the utility rate of return on investment allows equilibrium investment to equilibrate systematically at the level where (2) obtains: as the parameters of the model change slightly, the market rate of return can adjust to ensure that (2) remains satisfied. The proof of Proposition 1 (in the Appendix, which contains all proofs) makes this argument in greater detail.

Even though the aggregate supplies of output in the b and g states are equal in the intermediate equilibria under consideration, generically each agent's intermediate-period b endowment differs from his g endowment and therefore each agent trades. Each agent's market demands and utility therefore change as the price ratio takes on its various equilibrium values. This fact marks a difference relative to the portfolio inertia that comes with Dow–Werlang (1992) indeterminacy; see the discussion to come in Section 5.

For some combinations of parameters, the model has equilibria where (2) is not satisfied. To take an extreme example, $\sum_{i\in\mathcal{I}} e_g^i$ could be so large that even if all of the first-period good were invested, $\sum_{i\in\mathcal{I}}\overline{\theta}_1^i = \sum_{i\in\mathcal{I}} e_1^i$, the supply of goods in the g state would

[4]See the subsection "Two types of indeterminacy" below.

[5]In the intermediate period, agent i can optimize by maximizing $U^i(\overline{x}_1^i, x_b^i, x_g^i)$ subject to the reduced-form budget constraint $q_b x_b^i + q_g x_g^i = q_b(c\overline{\theta}_1^i + e_b^i) + q_g e_g^i$.

still outstrip the supply in the b state, $\sum_{i\in\mathcal{I}} e^i_g > \sum_{i\in\mathcal{I}}(c\overline{\theta}^i_1 + e^i_b)$. The same inequality would also hold in equilibrium if the marginal utility of first-period consumption were sufficiently high. Under either scenario, some agent i must consume more in the g state than in the b state; since this i evaluates consumption portfolios using the distribution $\overline{\pi}^i$ and therefore behaves locally like a classical utility maximizer, intermediate equilibria typically are locally unique.

Indeterminacy along the equilibrium path does not translate into indeterminacy of full equilibria. Suppose that $\overline{p}_b/\overline{p}_g$ is an equilibrium price ratio in the full intertemporal equilibrium that leads (2) to hold and that $\overline{p}_b/\overline{p}_g$ lies amid a continuum intermediate equilibrium price ratios. A slight change in q_b/q_g in the intermediate period from $\overline{p}_b/\overline{p}_q$ would change the ex post rate of return on investment. Had this change been anticipated, savings in the initial period would typically adjust in response, breaking the equality of the supply of goods in the b and g states. But then at least some agents must be consuming more in one state than in the other and hence relative prices must equal the unique ratio that supports these individuals' consumption bundles. This new supporting price ratio lies outside of the continuum of price ratios that support agents' indifference curves when second-period consumption is state-invariant; hence a scenario where a slight deviation from $\overline{p}_b/\overline{p}_g$ can still serve as an equilibrium price ratio in the full intertemporal equilibrium cannot, in fact, occur.

Two types of indeterminacy

The Bewley (2002) incomplete-preferences model of Knightian uncertainty, formulated as a general-equilibrium model in Rigotti and Shannon (2005), underscores the distinctive features of the indeterminacy under study. Unlike our model where indeterminacy in the intermediate period occurs only at isolated endowments, indeterminacy in the Bewley model persists under small endowment perturbations. This persistence makes it hard to argue that small parameter changes should have dramatic equilibrium consequences (the pattern we will see in Section 4). In the Bewley model, the equilibrium correspondence is typically continuous in the neighborhood of an equilibrium and, therefore, as Figure 2 makes clear, a small variation in endowments is consistent with price stability. One could posit that the prices selected from the equilibrium correspondence jump at some endowment point, but it would be hard to rationalize why markets happen to work in this way. Unlike Figure 1, those jumps are not hardwired in.

To make the comparison to Bewley explicit, we simplify our presentation of maximin ambiguity aversion. Let x^i denote a vector of uncertain consumption for agent i, for example (x^i_b, x^i_g) in the intermediate period of our model, and assume that each i maximizes $\min_{\pi\in\mathcal{P}^i} \mathbb{E}_\pi u^i(x^i)$ for some set of distributions $\mathcal{P}^i$ (where $\mathbb{E}_\pi$ denotes expectation calculated using the distribution π). If $x^{i\prime}$ is strictly preferred to x^i, then $x^{i\prime}$ must have strictly higher expected utility than x^i calculated using any distribution π^* that solves i's minimization problem when consumption equals x^i since

$$\mathbb{E}_{\pi^*} u^i(x^{i\prime}) \geq \min_{\pi\in\mathcal{P}^i} \mathbb{E}_\pi u^i(x^{i\prime}) > \min_{\pi\in\mathcal{P}^i} \mathbb{E}_\pi u^i(x^i) = \mathbb{E}_{\pi^*} u^i(x^i).$$

Therefore, $D_{x^i}\mathbb{E}_{\pi^*}u^i(x^i)$ supports the indifference curve of i that intersects x^i. So when x^i leads to multiple minimizing probabilities, say π^* and π^{**}, a continuum of normalized price vectors will support x^i (e.g., any $\alpha\pi^* + (1-\alpha)\pi^{**}$ with $\alpha \in [0,1]$ when x^i is state-invariant). A multiplicity of supporting prices can then lead to indeterminacy of equilibrium, as in the proof of Proposition 1. Alternatively, if a single π is minimizing, just one normalized price vector will support x^i, assuming that u^i is differentiable. Since multiple π's in $\mathcal{P}^i$ solve i's minimization problem only at unusual bundles x^i, indeterminacy is an exceptional event in the maximin model (although one that arises robustly) and hence the equilibrium correspondence will display the failure of lower hemicontinuity pictured in Figure 1.

In contrast, an agent i with Bewley preferences is defined to strictly prefer $(x_b^{i\prime}, x_g^{i\prime})$ to (x_b^i, x_g^i) if and only if

$$\mathbb{E}_\pi u^i(x^{i\prime}) > \mathbb{E}_\pi u^i(x^i) \quad \text{for all } \pi \in \mathcal{P}^i.$$

So a strict preference of a bundle $x^{i\prime}$ over x^i requires $x^{i\prime}$ to have higher expected utility than x^i calculated using any $\pi \in \mathcal{P}^i$. As a consequence, there is a continuum of supporting prices at *every* consumption bundle, not just at exceptional bundles. If, in addition, the agents in an economy have sets of supporting prices that robustly intersect, at some consumption profile, then indeterminacy is present: any vector in the intersection can serve as an equilibrium price vector (Rigotti and Shannon 2005). Under mild restrictions the intersection will persist as a continuous function of parameters and we arrive at the continuous correspondence pictured in Figure 2. The different continuity features of the equilibrium correspondences in Figures 1 and 2 explain why volatility appears readily in the maximin model: the failure of lower hemicontinuity in Figure 1 ensures that a small supply perturbation must lead to a large price response.

General technologies

The conclusion of Proposition 1 would not change if we were to let an initial-period investment θ_1^i generate a vector of outputs in the two states $(c_b\theta_1^i, c_g\theta_1^i)$. A switch of distributions and a discrete fall in the utility rate of return would then occur at the point where aggregate investment satisfies

$$\sum_{i\in\mathcal{I}}(c_b\theta_1^i + e_b^i) = \sum_{i\in\mathcal{I}}(c_g\theta_1^i + e_g^i).$$

As with (2), such investment levels can arise robustly. Assuming $\sum_{i\in\mathcal{I}} e_b^i \neq \sum_{i\in\mathcal{I}} e_g^i$, the above equality requires that $c_b \neq c_g$: investment must be able to change the economy's ratio of the state-contingent goods.

Indeterminacy with state-varying consumption

The key prerequisite for indeterminacy is that each agent i consumes a bundle at which a continuum of probabilities in $\mathcal{P}^i$ solves the minimization problem in (1) and where, therefore, indifference curves are kinked. The simplest way to satisfy this condition is for consumption to be state-invariant, in which case any $\pi \in \mathcal{P}^i$ is utility-minimizing. But if there are three or more states, then a continuum of solutions can also arise when agents are not perfectly hedged and consumption differs across states.

For example, let the set of states be $\{a, b, c\}$, let a first-period good be consumed or used to produce the state a good, set endowments to be $(e^i_a, e^i_b, e^i_c) = (0, 3, 4)$, and set probabilities and utility to be, respectively,

$$\mathcal{P}^i = \{(\pi_a, \pi_b, \pi_c) \geq 0 : \pi_a + 2\pi_b + 3\pi_c = 2 \text{ and } \pi_a + \pi_b + \pi_c = 1\}$$

$$U^i(x^i_1, x^i_a, x^i_b, x^i_c) = \min_{(\pi_a,\pi_b,\pi_c)\in\mathcal{P}^i} \sum_{s\in\{a,b,c\}} \pi_s(x^i_1 + x^i_s).$$

If the state b and c goods are nonproduced, then in an economy that consists only of agents of this type, each i must consume $(x^i_b, x^i_c) = (3, 4)$. If first-period investment leads to consumption x^i_a, then i's minimization problem in the intermediate period is $\min_{(\pi_a,\pi_b,\pi_c)\in\mathcal{P}^i} x^i_a\pi_a + 3\pi_b + 4\pi_c$, which has a unique solution $(\frac{1}{2}, 0, \frac{1}{2})$ when $x^i_a < 2$, a unique solution $(0, 1, 0)$ when $x^i_a > 2$, and a continuum of solutions $\{\alpha(\frac{1}{2}, 0, \frac{1}{2}) + (1-\alpha)(0, 1, 0) : 0 \leq \alpha \leq 1\}$ when $x^i_a = 2$. Given that utility is linear, these solution probabilities can serve as intermediate-period price vectors. Indeterminacy therefore occurs when $(x^i_a, x^i_b, x^i_c) = (2, 3, 4)$, i.e., when consumption varies by state. Moreover, if the return to a unit of period 1 investment is greater than 2 and e^i_1 is sufficiently large, the above agents will invest just the amount that leads to $x^i_a = 2$ and hence indeterminacy.[6]

4. Volatility of asset prices

We introduce a little uncertainty about productivity of the investment technology that translates the period 1 good into the state b good. The aggregate supplies of the state b and g goods then are almost never exactly equal in the intermediate period. Although indeterminacy therefore disappears, asset prices instead display nonnegligible variance no matter how small the investment uncertainty is. That the technological noise extinguishes price indeterminacy has the conceptual advantage that the resulting volatility tracks a real event (the outcome of the investment uncertainty) and hence cannot be interpreted as an artifact of attaching different equilibrium price vectors to different sunspots (see the discussion in Epstein and Wang 1994).

To see the cause of volatility, suppose an equilibrium approximately equalizes the aggregate supplies in the b and g states and that agents discover in the intermediate period that the technological noise has led to a relatively high level of output for the investment undertaken in period 1. Then the price of θ_b must assume a small value (because agents when calculating expected utilities will use a low π_b) as will the price of any asset whose payoff is weighted in favor of the b state. Conversely, a small level of

[6] It so happens in this example that indeterminacy does not appear robustly if there is more than one type of linear agent: with a one-dimensional set of minimizing probabilities for each agent, any nonempty intersection of the minimizing probabilities across agents will generically consist of a single point. One way to repair this problem is to introduce more states.

noise leads to a discretely higher price for θ_b. While this reasoning makes intuitive sense, there is still work to do; one must show that equilibrium investment falls somewhere in the band where, taking the noise into account, the events where the supply of the state b good is larger and smaller than the supply of the state g good both have nonnegligible probability.

As we will see, the variance of asset prices has a positive lower bound even as the investment uncertainty becomes trivial; the model therefore exhibits excess volatility. The volatility is, in fact, larger relative to fundamentals than, for example, in the Epstein and Schneider (2008) model of asset pricing under ambiguity. A smooth model of expected utility maximizers would, of course, behave differently; a small amount of technological noise would lead to only a small variation of asset prices.

Fix a model ω from Section 2 whose technology is described by the parameter $\tilde{c}$. For each positive integer t, the technology parameter now is a random variable c governed by a density h_t on a support $[\underline{c}_t, \overline{c}_t]$ (that is, $h_t(c) > 0 \Leftrightarrow c \in [\underline{c}_t, \overline{c}_t]$), where $\underline{c}_t \leq \tilde{c} \leq \overline{c}_t$ with at least one strict inequality. Agents learn the realization of c at the beginning of the intermediate period when they receive the proceeds of their first-period investment. Outside of the investment uncertainty, every feature of version t of ω coincides with that of ω. To let the noise shrink with t, we assume that $\overline{c}_t - \underline{c}_t \to 0$. We thus have a model with technological uncertainty for each t, which we call a *sequence of models with technological uncertainty*. We also say that the sequence *converges to* ω.

Consumptions x_b^i and x_g^i and asset demands θ_b^i and θ_g^i for agent i in version t are now functions from $[\underline{c}_t, \overline{c}_t]$ to $\mathbb{R}$ as are the prices p_b and p_g. With these new definitions in place and given $(p_b, p_g, e_1^i, e_b^i, e_g^i)$, an agent i's budget set in version t of the model is

$$\begin{aligned} B_t^i = \{(x_1^i, x_b^i, x_g^i, \theta_1^i, \theta_b^i, \theta_g^i) \in \mathbb{R}_+ \times \mathbb{R}_+^{[\underline{c}_t, \overline{c}_t]} \times \mathbb{R}_+^{[\underline{c}_t, \overline{c}_t]} \times \mathbb{R} \times \mathbb{R}^{[\underline{c}_t, \overline{c}_t]} \times \mathbb{R}^{[\underline{c}_t, \overline{c}_t]} : \\ x_1^i + \theta_1^i \leq e_1^i, \text{ and } \forall c \in [\underline{c}_t, \overline{c}_t], p_b(c)\theta_b^i(c) + p_g(c)\theta_g^i(c) \leq p_b(c)c\theta_1^i, \\ x_b^i(c) \leq e_b^i + \theta_b^i(c), x_g^i(c) \leq e_g^i + \theta_g^i(c)\}. \end{aligned}$$

We view h_t as the density of the objective distribution of c, since we are interested in the observable distribution of equilibria. Since we want to avoid any additional idiosyncratic ambiguity regarding the distribution of c that might be an independent source of volatility, we assume that all agents (unambiguously) believe that h_t governs c. For each agent i, utility is given by

$$\begin{aligned} V_t^i(x_1^i, x_b^i, x_g^i) &\equiv \int_{\underline{c}_t}^{\overline{c}_t} U^i(x_1^i, x_g^i(c), x_b^i(c)) h_t(c)\, dc \\ &= \int_{\underline{c}_t}^{\overline{c}_t} \min_{\pi \in \mathcal{P}^i} \left[\pi u^i(x_1^i, x_b^i(c)) + (1-\pi) u^i(x_1^i, x_g^i(c))\right] h_t(c)\, dc. \end{aligned}$$

The recursive structure of the above utility functions—in the initial period each i maximizes the expectation of the objective that i maximizes in the intermediate period—ensures dynamic consistency. So if a plan $(x_1^i, x_b^i, x_g^i, \theta_1^i, \theta_b^i, \theta_g^i)$ is optimizing in the initial period given the expected prices (p_b, p_g), then in the intermediate period, i will proceed to choose $(x_b^i, x_g^i, \theta_b^i, \theta_g^i)$ if (p_b, p_g), in fact, obtains.[7]

Definition 4. An *equilibrium for t* is $((x_1^i, x_b^i, x_g^i, \theta_1^i, \theta_b^i, \theta_g^i)_{i\in\mathcal{I}}, p_b, p_g)$ such that:

- For each $i \in \mathcal{I}$, $(x_1^i, x_b^i, x_g^i, \theta_1^i, \theta_b^i, \theta_g^i) \in B_t^i$ and $V_t^i(x_1^i, x_b^i, x_g^i) \geq V_t^i(x_1^{i\prime}, x_b^{i\prime}, x_g^{i\prime})$ for all $(x_1^{i\prime}, x_b^{i\prime}, x_g^{i\prime}, \theta_1^i, \theta_b^{i\prime}, \theta_g^{i\prime}) \in B_t^i$.
- $\sum_{i\in\mathcal{I}}(x_1^i + \theta_1^i) = \sum_{i\in\mathcal{I}} e_1^i$, $\sum_{i\in\mathcal{I}} \theta_1^i \geq 0$.
- For all $c \in [\underline{c}_t, \overline{c}_t]$: $\sum_{i\in\mathcal{I}} \theta_b^i(c) = \sum_{i\in\mathcal{I}} c\theta_1^i$, $\sum_{i\in\mathcal{I}} \theta_g^i(c) = 0$, $\sum_{i\in\mathcal{I}} x_b^i(c) = \sum_{i\in\mathcal{I}}(e_b^i + \theta_b^i(c))$, $\sum_{i\in\mathcal{I}} x_g^i(c) = \sum_{i\in\mathcal{I}}(e_g^i + \theta_g^i(c))$.

Given a sequence of models with technological uncertainty and an equilibrium for each t with prices $(p_b^t(c), p_g^t(c))_{c\in[\underline{c}_t,\overline{c}_t]}$, the variance of relative prices at t is given by

$$\mathrm{Var}\left(\frac{p_b^t(c)}{p_g^t(c)}\right) = \int_{\underline{c}_t}^{\overline{c}_t} \left(\frac{p_b^t(c)}{p_g^t(c)} - \mathbb{E}\left[\frac{p_b^t(c)}{p_g^t(c)}\right]\right)^2 h_t(c)\, dc,$$

where the expectation $\mathbb{E}$ is calculated using the density h_t. We define asset prices for the sequence to be *volatile in the limit* if, for any sequence of equilibrium prices $\langle(p_b^t, p_g^t)\rangle$, $\mathrm{Var}(\frac{p_b^t(c)}{p_g^t(c)})$ is bounded away from 0 for all t sufficiently large. When asset prices are volatile in the limit, the ratio of the variance of asset prices to the variance of the fundamental c increases without bound as $t \to \infty$. We focus on prices rather than on other endogenous variables out of tradition; the volatility of prices also leads the utility of agents to be volatile due to agents' trade in the intermediate period.

Let ω be a model from Section 2. A property of a sequence of models with technological uncertainty that converges to ω is *robust* if there is an open neighborhood Ω' of ω such that, for any $\omega' \in \Omega'$, the property holds for any sequence of models with technological uncertainty that converges to ω'. Since robustness requires a property to hold for *any* sequence of models that converges to any $\omega' \in \Omega'$, the h_t densities that govern c are unrestricted; we could instead incorporate h_t into the definition of a model and let properties be generic only if they hold for all densities with a sufficiently small support.

[7] Under the alternative modeling option where each i maximizes

$$\min_{\pi\in\mathcal{P}^i}\left[\int_{\underline{c}_t}^{\overline{c}_t} \left(\pi u^i(x_1^i, x_b^i(c)) + (1-\pi)u^i(x_1^i, x_g^i(c))\right)h_t(c)\, dc\right],$$

dynamic consistency would not obtain. There is, however, a "rectangular" set of probability measures $\mathcal{Q}^i$ defined on appropriate measurable subsets of $\{b, g\} \times [\underline{c}_t, \overline{c}_t]$ such that a maximin agent with utility u^i and the set of probabilities $\mathcal{Q}^i$ has the preferences represented by V_t^i and, in addition, is dynamically consistent. See Epstein and Schneider (2003).

Proposition 2. *There are sequences of models with technological uncertainty where it is robust for asset prices to be volatile in the limit.*

Our definition of equilibrium does not allow agents in the initial period to trade contracts that have payoffs contingent on the realization of c. This modeling decision is the simpler and more plausible path, but technically markets are incomplete. Fortunately the proof of Proposition 2 does not rely on this incompleteness. It is routine to define markets for the continuum of contracts that completeness would require; if we did so, then we could add "whether or not markets are complete" to the end of Proposition 2. The motive for this qualification is that a classical economy of expected utility maximizers with multiple equilibria might show nontrivial volatility in the presence of technological noise when markets are incomplete: the realization of c could effectively serve as a sunspot that determines which equilibrium plays out, thus allowing a randomization over the multiple equilibria. When markets are complete, c cannot serve as a sunspot and then there is an unambiguous volatility difference between ambiguity-averse and expected-utility economies. Alternatively, we could maintain the sharp divide between the two types of economies if we consider only expected-utility economies with a single equilibrium.

5. Conclusion

The seemingly unusual event that the aggregate supply of output falls into a configuration that generates indeterminacy occurs systematically with ambiguity aversion and intertemporal production. Ambiguity aversion introduces a discontinuity in the rate of return on investment at just the points where investment results in indeterminacy; the discontinuity ensures that these particular investment levels arise robustly. That indeterminacy occurs only at specific output supplies gives it economic potency, which we have illustrated by showing that asset prices display a large reaction to small random events.

The indeterminacy considered here is a direct descendant of Dow–Werlang (1992), but there is an important difference. In Dow and Werlang, agents are implicitly endowed with the same quantity of goods in the b and g states and stick to that endowment, declining to buy or sell an asset over a range of the asset's price. As a consequence, asset price variations do not affect demand or utility. In this paper, agents typically have to trade in the intermediate period so as to reach their utility-maximizing second-period consumption bundles. Since agents have to trade in the intermediate period, any variation in equilibrium prices in the intermediate period changes their demands and utilities: if, say, the price of the state b good rises slightly, then an agent who previously purchased that good in the intermediate period now buys less of it and is worse off.

With a different asset structure, we could get Dow–Werlang indeterminacy and portfolio inertia. For example, suppose that agents in period 1 can buy not only productive shares in the firm, but also assets that deliver claims on the state b and g goods. Each agent then is able in his first-period trades to buy rights to his second-period consumption bundle. With these demands, agents do not need to trade further in the intermediate period, but if intermediate-period markets are open, then agents can refrain from trade over a continuum of prices, as in Dow and Werlang.

Appendix: Proofs

Proof of Proposition 1. For each $i \in \mathcal{I}$, fix some arbitrary u^i that satisfies our assumptions and $(\overline{x}_1^i, \overline{x}_2^i) \gg 0$, and set some $c > 0$. We build a model and equilibrium where agent i has u^i and consumes $(\overline{x}_1^i, \overline{x}_2^i)$.

Agent i's optimization problem can be solved by maximizing $U^i(x_1^i, x_b^i, x_g^i)$ subject to the unified budget constraint $cp_b x_1^i + p_b x_b^i + p_g x_g^i \le cp_b e_1^i + p_b e_b^i + p_g e_g^i$. If $(\widehat{x}_1^i, \widehat{x}_2^i, \widehat{\lambda}^i) \ge 0$ satisfies the first order conditions

$$cp_b x_1^i + (p_b + p_g) x_2^i = cp_b e_1^i + p_b e_b^i + p_g e_g^i \tag{3}$$

$$D_{x_1^i} u^i(x_1^i, x_2^i) - \lambda^i c p_b = 0 \tag{4}$$

$$D_{x_2^i} u^i(x_1^i, x_2^i) - \lambda^i (p_b + p_g) = 0 \tag{5}$$

$$\underline{\pi}^i D_{x_2^i} u^i(x_1^i, x_2^i) - \lambda^i p_b < 0 \tag{6}$$

$$\overline{\pi}^i D_{x_2^i} u^i(x_1^i, x_2^i) - \lambda^i p_b > 0 \tag{7}$$

$$(1 - \overline{\pi}^i) D_{x_2^i} u^i(x_1^i, x_2^i) - \lambda^i p_g < 0 \tag{8}$$

$$(1 - \underline{\pi}^i) D_{x_2^i} u^i(x_1^i, x_2^i) - \lambda^i p_g > 0, \tag{9}$$

then

$$L(x_1^i, x_b^i, x_g^i) = U^i(x_1^i, x_b^i, x_g^i) - \lambda(cp_b x_1^i + p_b x_b^i + p_g x_g^i - cp_b e_1^i - p_b e_b^i - p_g e_g^i)$$

is maximized at $(x_1^i, x_b^i, x_g^i) = (\widehat{x}_1^i, \widehat{x}_2^i, \widehat{x}_2^i)$ and hence this vector solves i's optimization problem. Note in this regard that conditions (6)–(9) state that the right (left) hand side derivatives of L with respect to x_b and x_g are negative (positive), and one may use (5)–(9) to check that L is then maximized with respect to any direction of change in (x_b^i, x_g^i).

For appropriate choices of $(\lambda^i, e_1^i, e_b^i, e_g^i, \underline{\pi}^i, \overline{\pi}^i, p_b, p_g)$, $(\overline{x}_1^i, \overline{x}_2^i)$ satisfies (3)–(9). For instance, set $e_g^i = \overline{x}_2^i$, let $e_b^i < e_g^i$ be arbitrary, and set $e_1^i = \overline{x}_1^i + (1/c)(e_g^i - e_b^i)$. Then (3) is satisfied regardless of p_b or p_g. Set an arbitrary $\overline{p}_b$ and set $\overline{\lambda}^i$ to satisfy (4) when $(x_1^i, x_2^i) = (\overline{x}_1^i, \overline{x}_2^i)$. Given $(\overline{x}_1^i, \overline{x}_2^i, \overline{\lambda}^i, \overline{p}_b)$, set $\overline{p}_g$ to satisfy (5). Finally, let $\widehat{\pi}^i$ be defined by $\widehat{\pi}^i D_{x_2^i} u^i(\overline{x}_1^i, \overline{x}_2^i) - \overline{\lambda}^i \overline{p}_b = 0$. Then, by (5), $(1 - \widehat{\pi}^i) D_{x_2^i} u^i(\overline{x}_1^i, \overline{x}_2^i) - \overline{\lambda}^i \overline{p}_g = 0$. Hence, by setting $\underline{\pi}^i < \widehat{\pi}^i < \overline{\pi}^i$, inequalities (6)–(9) are satisfied when $(x_1^i, x_2^i, \lambda^i, p_b, p_g) = (\overline{x}_1^i, \overline{x}_2^i, \overline{\lambda}^i, \overline{p}_b, \overline{p}_g)$.

Let asset holdings be given by $\overline{\theta}_1^i = e_1^i - \overline{x}_1^i$, $\overline{\theta}_g^i = 0$, and $\overline{\theta}_b^i = \overline{x}_b^i - e_b^i$ for each $i \in \mathcal{I}$. It is easy to see that the market-clearing conditions in Definition 1 are then satisfied and so $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i \in \mathcal{I}}, \overline{p}_b, \overline{p}_g)$ is, in fact, an equilibrium.

Given this equilibrium, $((\overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i \in \mathcal{I}}, q_b = \overline{p}_b, q_g = \overline{p}_g)$ is an intermediate equilibrium. This intermediate equilibrium is indeterminate as is any intermediate equilibrium derived from an equilibrium $((\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i, \overline{\theta}_1^i, \overline{\theta}_b^i, \overline{\theta}_g^i)_{i \in \mathcal{I}}, \overline{p}_b, \overline{p}_g)$, where $\overline{x}_g^i =$

$\overline{x}^i_b$ for all $i \in \mathcal{I}$ and (6)–(9) are satisfied. To see this, observe that the unified budget constraint for agent i that applies in the intermediate period is $q_b x^i_b + q_g x^i_g = q_b(e^i_b + c\overline{\theta}^i_1) + q_g e^i_g$. As long as q_b/q_m is sufficiently near to $\overline{p}_b/\overline{p}_g$, the x^i_2 that satisfies $(q_b + q_g)x^i_2 = q_b(e^i_b + c\overline{\theta}^i_1) + q_g e^i_g$ is i's equilibrium choice of both x_b and x_g.[8] Agent i's intermediate equilibrium choices of (θ^i_b, θ^i_g) then follow from this choice of $x_b = x_g$. Due to the fact that $\sum_{i\in\mathcal{I}}(e^i_b + c\overline{\theta}^i_1) = \sum_{i\in\mathcal{I}} e^i_g$, these demands are consistent with market clearing.

Let $\overline{\omega}$ denote the model $((u^i, \underline{\pi}^i, \overline{\pi}^i, e^i_1, e^i_b, e^i_g)_{i\in\mathcal{I}}, c)$ we have constructed. Since each agent i in the equilibrium we have identified for $\overline{\omega}$ consumes $\overline{x}^i_1 = e^i_1(\overline{\omega}) + (1/c)(e^i_b(\overline{\omega}) - e^i_g(\overline{\omega}))$, $\overline{x}^i_b = \overline{x}^i_g = e^i_g(\overline{\omega})$ (given access to the technology, each i could consume without trade), it follows that $\overline{\omega}$ has a unique equilibrium. We omit the standard argument that if there were more than one equilibrium for some $\widehat{\omega}$ in every open $\widehat{\Omega}$ containing $\overline{\omega}$, then $\overline{\omega}$ would have an additional equilibrium too.

We show that there is an open set of models $\overline{\Omega}$ with $\overline{\omega} \in \overline{\Omega}$ and a continuous function f from $\overline{\Omega}$ to an appropriately normalized set of prices such that $f(\omega)$ is an equilibrium price vector for $\omega \in \overline{\Omega}$ and $f(\overline{\omega}) = (\overline{p}_b, \overline{p}_g)$. As we will see, (3)–(9) will remain satisfied on a small enough neighborhood of $\overline{\omega}$ and hence each i in equilibrium will consume the same quantities in states b and g. Indeterminacy of the intermediate equilibria therefore continues to obtain. Since equilibrium is unique in a neighborhood of $\overline{\omega}$, we conclude that it is robust for all equilibria to be indeterminate along the equilibrium path. Below, we indicate the dependence of a parameter on the model ω by the notation $e^i_1(\omega)$, $u^i(\omega)$, and so forth.

Given the negative definiteness of $D^2u^i(x^i_1, x^i_2)$ and our assumption that if $\omega_n \to \overline{\omega}$ then $Du^i(\omega_n) \to Du^i$ on any compact K, it is an exercise in demand theory to use the implicit function theorem to show that (3)–(5) can be solved locally for $(x^i_1, x^i_2, \lambda^i)$ as functions of $(p_b, p_g; \omega)$ in a neighborhood of $(\overline{p}_b, \overline{p}_g; \overline{\omega})$, thus producing continuous functions $x^i_1(p_b, p_g; \omega)$, $x^i_2(p_b, p_g; \omega)$, and $\lambda^i(p_b, p_g; \omega)$ that are continuously differentiable in (p_b, p_g) and where $(x^i_1(\overline{p}_b, \overline{p}_g; \overline{\omega}), x^i_2(\overline{p}_b, \overline{p}_g; \overline{\omega}), \lambda^i(\overline{p}_b, \overline{p}_g; \overline{\omega})) = (\overline{x}^i_1, \overline{x}^i_2, \overline{\lambda}^i)$.[9] Since (6)–(9) are inequalities, these conditions remain satisfied for all $(p_b, p_g; \omega)$ in some open neighborhood of $(\overline{p}_b, \overline{p}_g; \overline{\omega})$. Hence $(x^i_1(p_b, p_g; \omega), x^i_2(p_b, p_g; \omega), x^i_2(p_b, p_g; \omega))$

[8]Beyond the budget constraint, the other first order conditions that must be satisfied when $x_b = x_g$ are

$$
\begin{aligned}
D_{x^i_2} u^i(\overline{x}^i_1, x^i_2) - \lambda^i(q_b + q_g) &= 0 \\
\underline{\pi}^i D_{x^i_2} u^i(\overline{x}^i_1, x^i_2) - \lambda^i q_b &< 0 \\
\overline{\pi}^i D_{x^i_2} u^i(\overline{x}^i_1, x^i_2) - \lambda^i q_b &> 0 \\
(1 - \overline{\pi}^i) D_{x^i_2} u^i(\overline{x}^i_1, x^i_2) - \lambda^i q_g &< 0 \\
(1 - \underline{\pi}^i) D_{x^i_2} u^i(\overline{x}^i_1, x^i_2) - \lambda^i q_g &> 0.
\end{aligned}
$$

Given the x^i_2 determined by the budget constraint, the first condition above determines λ^i. Since the remaining conditions are inequalities, they remain satisfied for q_b/q_m near $\overline{p}_b/\overline{p}_g$.

[9]A topological version of the implicit function theorem is needed (see Schwartz 1967).

solves (3)–(9) and comprises agent i's optimizing demands on an open neighborhood of $(\overline{p}_b, \overline{p}_g; \overline{\omega})$.

By homogeneity, we can renormalize and henceforth constrain p_b to equal 1. So $\overline{\overline{p}}_g = \overline{p}_g/\overline{p}_b$ is the equilibrium price for the state g good at $\overline{\omega}$. Now suppose that $\sum_{i\in\mathcal{I}} D_{p_g} x_2^i(1, \overline{\overline{p}}_g; \overline{\omega}) \neq 0$. Another application of the implicit function theorem implies there is a continuous function $p_g(\omega)$, defined on an open $\Omega' \subset \Omega$ that contains $\overline{\omega}$, such that $p_g(\overline{\omega}) = \overline{\overline{p}}_g$ and

$$\sum_{i\in\mathcal{I}} \bigl(x_2^i(1, p_g(\omega); \omega) - e_g^i(\omega)\bigr) = 0. \tag{10}$$

(So the previously mentioned function $f : \overline{\Omega} \to \{1\} \times \mathbb{R}_+$ is defined by $f(\omega) = (1, p_g(\omega))$.) To check that satisfying (10) leads to a full equilibrium, observe that by the budget constraints that define B_i, asset demands, which are now also functions of $(p_b, p_g; \omega)$, must be given by

$$\theta_1^i(1, p_g(\omega); \omega) = e_1^i(\omega) - x_1^i(1, p_g(\omega); \omega) \tag{11}$$

$$\theta_b^i(1, p_g(\omega); \omega) = x_2^i(1, p_g(\omega); \omega) - e_b^i(\omega) \tag{12}$$

$$\theta_g^i(1, p_g(\omega); \omega) = x_2^i(1, p_g(\omega); \omega) - e_g^i(\omega). \tag{13}$$

Given (10), summing the equality (13) over i gives $\sum_{i\in\mathcal{I}} \theta_g^i(1, p_g(\omega); \omega) = 0$, so the equilibrium market-clearing requirements for both θ_g in the intermediate period and x_g in the final period are met. Summing the equalities (11) and (12) over i implies that the x_1 market in the first period and the x_b market in the final period clear. Finally, by summing the equality (3) over i, and using (10) and the equalities (11) and (12), we have $\sum_{i\in\mathcal{I}} \theta_b^i(1, p_g(\omega); \omega) = \sum_{i\in\mathcal{I}} c\theta_1^i(1, p_g(\omega); \omega)$, the market-clearing requirement for θ_b. Since $p_b = 1$, $p_g(\omega)$, $x_1^i(1, p_g(\omega); \omega)$, $x_2^i(1, p_g(\omega); \omega)$, and $\lambda^i(1, p_g(\omega); \omega)$ allow conditions (3)–(9) to remain satisfied for ω in an open neighborhood of $\overline{\omega}$, our earlier indeterminacy argument implies that

$$\Bigl(\bigl(x_2^i(1, p_g(\omega); \omega), x_2^i(1, p_g(\omega); \omega), \theta_b^i(1, p_g(\omega); \omega), \theta_g^i(1, p_g(\omega); \omega)\bigr)_{i\in\mathcal{I}},$$
$$q_b = 1, q_g = p_g(\omega)\Bigr)$$

is indeterminate for ω in the same neighborhood.

It remains to show that $\sum_{i\in\mathcal{I}} D_{p_g} x_g^i(1, \overline{\overline{p}}_g; \overline{\omega}) \neq 0$. Consider the supplementary problem in the model $\overline{\omega}$ of maximizing $u^i(x_1^i, x_2^i)$ subject to $cp_b x_1^i + (p_b + p_g)x_2^i \leq I$ and let $m_2^i(cp_b, p_b + p_g, I)$ denote the solution x_2^i to this problem, i.e., the Marshallian demand for good 2. Since i's income is $ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i$ at $\overline{\omega}$ (now dropping dependence on ω from our notation), differentiation gives

$$D_{p_g} x_g^i(1, \overline{\overline{p}}_g; \overline{\omega}) = D_2 m_2^i(c, 1 + \overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i)$$
$$+ D_3 m_2^i(c, 1 + \overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i) e_g^i.$$

Letting $h_2^i(cp_b, p_b + p_g, u)$ be the agent's Hicksian demand for good 2, the Slutsky equation states that

$$\begin{aligned} &D_2 m_2^i(c, 1+\overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i) \\ &\quad = D_2 h_2^i(c, 1+\overline{\overline{p}}_g, U^i(\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i)) \\ &\qquad\qquad - m_2^i(c, 1+\overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i) D_3 m_2^i(c, 1+\overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i). \end{aligned}$$

Since $e_g^i = m_2^i(c, 1+\overline{\overline{p}}_g, ce_1^i + e_b^i + \overline{\overline{p}}_g e_g^i)$, we have $D_{p_g} x_g^i(1, \overline{\overline{p}}_g; \overline{\omega}) = D_2 h_2^i(c, 1+\overline{\overline{p}}_g, U^i(\overline{x}_1^i, \overline{x}_b^i, \overline{x}_g^i))$, which, by classical demand theory, is strictly negative. Hence

$$\sum_{i\in\mathcal{I}} D_{p_g} x_g^i(1, \overline{\overline{p}}_g; \overline{\omega}) \neq 0.$$

□

Proof of Proposition 2. In Part 1, we construct an agent γ whose utility is maximized only when savings of the first-period good are at the level, $e_1^\gamma - \overline{x}_1^\gamma$, that leads second-period consumption in states b and g to be equal. Then in Part 2 we show that if there were a set of I agents, each with characteristics near that of γ, and if equilibrium prices were not volatile in the limit, then there would have to be a solution to the problem of maximizing the sum of I copies of the γ utility that differs from each agent saving $e_1^\gamma - \overline{x}_1^\gamma$.

Part 1. Let u^γ be a utility that meets our assumptions such that

$$D_{x_1} u^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma) = \widehat{c}\widehat{\pi} D_{x_2} u^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma) \tag{14}$$

is satisfied for some $(\overline{x}_1^\gamma, \overline{x}_2^\gamma) \gg 0$, $\widehat{\pi} \in (0,1)$, and $\widehat{c} > 0$. Let $(e_1^\gamma, e_b^\gamma, e_g^\gamma)$ satisfy $e_b^\gamma + \widehat{c}(e_1^\gamma - \overline{x}_1^\gamma) = e_g^\gamma = \overline{x}_2^\gamma$ and $e_1^\gamma > \overline{x}_1^\gamma$, and let $E_\pi u^\gamma(x_1^\gamma)$, where $\pi \in [0,1]$, denote $\pi u^\gamma(x_1^\gamma, e_b^\gamma + \widehat{c}(e_1^\gamma - x_1^\gamma)) + (1-\pi) u^\gamma(x_1^\gamma, e_g^\gamma)$. Then (14) is the first order condition that shows that $\overline{x}_1^\gamma$ solves the problem $\max_{x_1^\gamma} E_{\widehat{\pi}} u^\gamma(x_1^\gamma)$ subject to $x_1^\gamma \in [0, e_1^\gamma]$. Then for $x_1^\gamma \in [0, e_1^\gamma]$,

$$\min_{\pi\in[\underline{\pi}^\gamma, \overline{\pi}^\gamma]} E_\pi u^\gamma(\overline{x}_1^\gamma) = E_{\widehat{\pi}} u^\gamma(\overline{x}_1^\gamma) \geq E_{\widehat{\pi}} u^\gamma(x_1^\gamma) \geq \min_{\pi\in[\underline{\pi}^\gamma, \overline{\pi}^\gamma]} E_\pi u^\gamma(x_1^\gamma),$$

where the equality follows from the fact that $e_b^\gamma + \widehat{c}(e_1^\gamma - \overline{x}_1^\gamma) = e_g^\gamma$. Hence $\overline{x}_1^\gamma$ solves $\max_{x_1^\gamma}(\min_{\pi\in[\underline{\pi}^\gamma, \overline{\pi}^\gamma]} E_\pi u^\gamma(x_1^\gamma))$ subject to $x_1^\gamma \in [0, e_1^\gamma]$.

We now let c be uncertain and note two easily confirmed facts, omitting their simple proofs. First, if $\langle(e_1^\gamma(n), e_b^\gamma(n), e_g^\gamma(n), \underline{c}(n), \overline{c}(n))\rangle$ converges to $(e_1^i, e_b^i, e_g^i, \widehat{c}, \widehat{c})$ and, for each n, h_n is an arbitrary density on $[\underline{c}(n), \overline{c}(n)]$, then the solution to the single-agent problem

$$\begin{aligned} \max_{x_1^i \geq 0} \int_{\underline{c}(n)}^{\overline{c}(n)} \min_{\pi\in[\underline{\pi}^\gamma, \overline{\pi}^\gamma]} \big(&\pi u^\gamma(x_1^\gamma, e_b^\gamma(n) + c(e_1^\gamma(n) - x_1^\gamma)) \\ &+ (1-\pi) u^\gamma(x_1^\gamma, e_g^\gamma(n))\big) h_n(c)\, dc \end{aligned} \tag{15}$$

subject to $x_1^\gamma \leq e_1^\gamma(n)$ converges to $\overline{x}_1^\gamma$. Second, it follows that the solution to the problem of maximizing the sum of I copies of the utility that we have defined,

$$\max_{(x_1^1,\dots,x_1^I)\geq 0} \sum_{k=1}^{I} \biggl(\int_{\underline{c}(n)}^{\overline{c}(n)} \min_{\pi \in [\underline{\pi}^\gamma, \overline{\pi}^\gamma]} \bigl(\pi u^\gamma (x_1^k, e_b^\gamma(n) + c(e_1^\gamma(n) - x_1^k)) + (1-\pi) u^\gamma (x_1^k, e_g^\gamma(n)) \bigr) h_n(c)\, dc \biggr) \tag{16}$$

subject to $\sum_{k=1}^I x_1^k \leq \sum_{k=1}^I e_1^\gamma(n)$, converges to a vector $(x_1^1, \dots, x_1^I)$ equal to I copies of $\overline{x}_1^\gamma$.

Part 2.

Lemma 1. *Let $\mathcal{P}^k = [\underline{\pi}^k, \overline{\pi}^k]$ for $k \in \mathcal{I}$ and suppose (i) $\underline{\rho}, \overline{\rho} \in (0,1)$ satisfy $\underline{\rho} \geq \underline{\pi}^k$ and $\overline{\rho} \leq \overline{\pi}^k$ for all $k \in \mathcal{I}$, and (ii) $((\overline{x}_1^k, \overline{x}_g^k, \overline{x}_b^k, \overline{\theta}_1^k, \overline{\theta}_g^k, \overline{\theta}_b^k)_{k\in\mathcal{I}}, \overline{p}_b, \overline{p}_g)$ is an equilibrium for some version t of some model. Then there is a subset $C \subset [\underline{c}_t, \overline{c}_t]$ with Lebesgue measure $\overline{c}_t - \underline{c}_t$ such that, for any $c \in C$, $\sum_{k\in\mathcal{I}}(e_b^k + c(e_1^k - \overline{x}_1^k)) > \sum_{k\in\mathcal{I}} e_g^k$ (resp. $\sum_{k\in\mathcal{I}}(e_b^k + c(e_1^k - \overline{x}_1^k)) < \sum_{k\in\mathcal{I}} e_g^k$) implies*

$$\frac{\overline{p}_b(c)}{\overline{p}_g(c)} \leq \frac{\underline{\rho}}{1-\underline{\rho}} \quad \biggl(\textit{resp. } \frac{\overline{p}_b(c)}{\overline{p}_g(c)} \geq \frac{\overline{\rho}}{1-\overline{\rho}}\biggr).$$

Proof. To avoid vacuities, assume

$$\exists c \in [\underline{c}_t, \overline{c}_t] \quad \text{such that} \quad \sum_{k\in\mathcal{I}}(e_b^k + c(e_1^k - \overline{x}_1^k)) \neq \sum_{k\in\mathcal{I}} e_g^k. \tag{17}$$

For each agent k, there must be a $C^k \subset [\underline{c}_t, \overline{c}_t]$ of measure $\overline{c}_t - \underline{c}_t$ such that, for $c \in C^k$, $(\overline{x}_b^k(c), \overline{x}_g^k(c))$ maximizes $U^k(x_1^k, x_b^k(c), x_g^k(c))$ subject to $\overline{p}_b(c) x_b^k(c) + \overline{p}_g(c) x_g^k(c) \leq \overline{p}_b(c)(c\overline{\theta}_1^k + e_b^k) + \overline{p}_g(c) e_g^k$, $(x_b^k(c), x_g^k(c)) \geq 0$. Set

$$C = \bigcap_{k\in\mathcal{I}} C^k \setminus \biggl\{ c \in [\underline{c}_t, \overline{c}_t] \colon \sum_{k\in\mathcal{I}}(e_b^k + c(e_1^k - \overline{x}_1^k)) = \sum_{k\in\mathcal{I}} e_g^k \biggr\}.$$

Given (17), the last set above has at most one element and hence C has measure $\overline{c}_t - \underline{c}_t$. Now suppose that $\sum_{k\in\mathcal{I}}(e_b^k + c(e_1^k - \overline{x}_1^k)) > \sum_{k\in\mathcal{I}} e_g^k$ for some $c \in C$. Then there must be a $j \in \mathcal{I}$ for which $\overline{x}_b^j(c) > \overline{x}_g^j(c)$. Hence

$$\frac{\overline{p}_b(c)}{\overline{p}_g(c)} \leq \frac{D_{x_b^j} U^j(\overline{x}_1^j, \overline{x}_b^j(c), \overline{x}_g^j(c))}{D_{x_g^j} U^j(\overline{x}_1^j, \overline{x}_b^j(c), \overline{x}_g^j(c))}$$

with strict inequality possible when $\overline{x}_g^i(c) = 0$. For any $(x_1^k, x_2^k) \in \mathbb{R}_+^2$ and any $k \in \mathcal{I}$,

$$\frac{D_{x_b^k}^+ U^k(x_1^k, x_2^k, x_2^k)}{D_{x_g^k}^- U^k(x_1^k, x_2^k, x_2^k)} = \frac{\underline{\pi}^k}{1-\underline{\pi}^k},$$

where the superscripts $+$ and $-$ indicate right and left derivatives, respectively. Since the concavity of u^j gives

$$\frac{D_{x_b^j} U^j(\overline{x}_1^j, \overline{x}_b^j(c), \overline{x}_g^j(c))}{D_{x_g^j} U^j(\overline{x}_1^j, \overline{x}_b^j(c), \overline{x}_g^j(c))} \leq \frac{D^+_{x_b^j} U^j(x_1^j, x_2^j, x_2^j)}{D^-_{x_g^j} U^j(x_1^j, x_2^j, x_2^j)}$$

and since $\underline{\pi}^k/(1-\underline{\pi}^k) \leq \underline{\rho}/(1-\underline{\rho})$, we have the desired conclusion. The case

$$\sum_{k\in\mathcal{I}} (e_b^k + c(e_1^k - \overline{x}_1^k)) < \sum_{k\in\mathcal{I}} e_g^k$$

is similar. ◁

Consider an arbitrary sequence $\langle\omega_n\rangle$, where

$$\omega_n = \big((u^k(n), \underline{\pi}^k(n), \overline{\pi}^k(n), e_1^k(n), e_b^k(n), e_g^k(n))_{k\in\mathcal{I}}, \underline{c}(n), \overline{c}(n), h_n\big),$$

such that, for each $k \in \mathcal{I}$, $u^k(n) \to u^\gamma$ uniformly on compact sets,

$$(\underline{\pi}^k(n), \overline{\pi}^k(n), e_1^k(n), e_b^k(n), e_g^k(n), \underline{c}(n), \overline{c}(n)) \to (\underline{\pi}^\gamma, \overline{\pi}^\gamma, e_1^\gamma, e_b^\gamma, e_g^\gamma, \widehat{c}, \widehat{c}),$$

and where $\overline{c}(n) > \underline{c}(n)$ for each n. To prove the proposition, it is sufficient to show, for any such $\langle\omega_n\rangle$ and any corresponding sequence of equilibria

$$\big\langle((x_1^k(n), x_g^k(n), x_b^k(n), \theta_1^k(n), \theta_g^k(n), \theta_b^k(n))_{k\in\mathcal{I}}, p_b(n), p_g(n))\big\rangle$$

for $\langle\omega_n\rangle$, that $\mathrm{Var}(p_b(n)/p_g(n))$ is bounded away from 0 for all n sufficiently large. Define

$$R(n) = \frac{\mu(\{c \in [\underline{c}(n), \overline{c}(n)] : \sum_{k\in\mathcal{I}} e_g^k(n) > \sum_{k\in\mathcal{I}}(e_b^k(n) + c(e_1^k(n) - x_1^k(n)))\})}{\mu(\{c \in [\underline{c}(n), \overline{c}(n)] : \sum_{k\in\mathcal{I}} e_g^k(n) < \sum_{k\in\mathcal{I}}(e_b^k(n) + c(e_1^k(n) - x_1^k(n)))\})},$$

where μ is Lebesgue measure. Setting $\underline{\rho}$ and $\overline{\rho}$ so that $\overline{\pi}^\gamma > \overline{\rho} > \widehat{\pi} > \underline{\rho} > \underline{\pi}^\gamma$, we have $\underline{\pi}^k(n) \leq \underline{\rho}$ and $\overline{\pi}^k(n) \geq \overline{\rho}$ for all k and all large n. Hence Lemma 1 implies that if there is no subsequence with either $R(n) \to \infty$ or $R(n) \to 0$, then $\mathrm{Var}(p_b(n)/p_g(n))$ will be bounded away from 0 for all n sufficiently large.

Suppose, to the contrary, that there is a subsequence with $R(n) \to \infty$ (we omit the similar case $R(n) \to 0$). Then Lemma 1 implies that the ratio

$$\mu\left(\left\{c \in [\underline{c}(n), \overline{c}(n)] : \frac{p_b(n)(c)}{p_g(n)(c)} < \frac{\overline{\rho}}{1-\overline{\rho}}\right\}\right) \Big/ \mu\left(\left\{c \in [\underline{c}(n), \overline{c}(n)] : \frac{p_b(n)(c)}{p_g(n)(c)} \geq \frac{\overline{\rho}}{1-\overline{\rho}}\right\}\right)$$

converges to 0 along the subsequence. Observe that the γ agent from Part 1 (with the utility u^γ, endowments $(e_1^\gamma, e_b^\gamma, e_g^\gamma)$, and $\underline{\pi}^\gamma < \widehat{\pi} < \overline{\pi}^\gamma$) will, if c equals $\widehat{c}$ with certainty, consume the quantities $(x_1^\gamma, x_b^\gamma, x_g^\gamma) = (\overline{x}_1^\gamma, \overline{x}_2^\gamma, \overline{x}_2^\gamma)$ when (p_b, p_g) satisfies

$$\frac{D_{x_1} u^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma)}{D_{x_2} u^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma)} = \frac{\widehat{c} p_b}{p_b + p_g} \tag{18}$$

(see (4) and (5)). Given (14), the unique ratio $\frac{p_b}{p_g}$ such that (p_b, p_g) satisfies (18) is $\widehat{\pi}/(1-\widehat{\pi})$, which lies in the interval $(\underline{\pi}^\gamma/(1-\underline{\pi}^\gamma), \overline{\pi}^\gamma/(1-\overline{\pi}^\gamma))$. Suppose we face the same agent with some price ratio $\widetilde{p}_b/\widetilde{p}_g \in (\widehat{\pi}/(1-\widehat{\pi}), \overline{\rho}/(1-\overline{\rho}))$. Since $(\overline{x}_1^\gamma, \overline{x}_2^\gamma)$ is still in the unified budget set that constrains x_b^γ to equal x_g^γ,

$$B(\widetilde{p}_b, \widetilde{p}_g, \widehat{c}) \equiv \{(x_1^\gamma, x_2^\gamma) \in \mathbb{R}_+^2 : \widehat{c}\widetilde{p}_b x_1^\gamma + (\widetilde{p}_b + \widetilde{p}_g)x_2^\gamma \le \widehat{c}\widetilde{p}_b e_1^\gamma + \widetilde{p}_b e_b^\gamma + \widetilde{p}_g e_g^\gamma\},$$

and since u^γ is differentiable, there must be a $(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma) \in B(\widetilde{p}_b, \widetilde{p}_g, \widehat{c})$ such that $U^\gamma(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma, \widetilde{x}_2^\gamma) > U^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma, \overline{x}_2^\gamma)$.[10] In fact, $(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma) \in B(p_b, p_g, \widehat{c})$ for any $p_b/p_g > \widetilde{p}_b/\widetilde{p}_g$. Given this and since $\underline{c}(n) \to \widehat{c}$ and $\overline{c}(n) \to \widehat{c}$, for all n sufficiently large $(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma) \in B(p_b, p_g, c)$ for any $c \in [\underline{c}(n), \overline{c}(n)]$ and any (p_b, p_g) with $p_b/p_g \ge \overline{\rho}/(1-\overline{\rho})$. Therefore, since $R(n) \to \infty$, for each $k \in \mathcal{I}$ there is a N such that for $n > N$ the utility level achieved by k in the equilibrium of ω_n satisfies

$$\int_{\underline{c}(n)}^{\overline{c}(n)} U_n^k(x_1^k(n), x_b^k(n)(c), x_g^k(n)(c))h_n(c)\,dc \ge U_n^k(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma, \widetilde{x}_2^\gamma).$$ [11]

Since $u^k(n)(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma) \to u^\gamma(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma)$, $U_n^k(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma, \widetilde{x}_2^\gamma) \to U^\gamma(\widetilde{x}_1^\gamma, \widetilde{x}_2^\gamma, \widetilde{x}_2^\gamma)$ and hence there is a $\varepsilon > 0$ such that, for all n sufficiently large,

$$\int_{\underline{c}(n)}^{\overline{c}(n)} U_n^k(x_1^k(n), x_b^k(n)(c), x_g^k(n)(c))h_n(c)\,dc \ge U^\gamma(\overline{x}_1^\gamma, \overline{x}_2^\gamma, \overline{x}_2^\gamma) + \varepsilon. \tag{19}$$

Since $u^k(n) \to u^\gamma$ uniformly on compact sets, $U_n^k \to U^\gamma$ uniformly on a compact set that contains $\{(x_1^k(n), x_b^k(n)(c), x_g^k(n)(c)) : n \ge 1, \underline{c}(n) \le c \le \overline{c}(n)\}$. Hence

$$\int_{\underline{c}(n)}^{\overline{c}(n)} \big[U_n^k(x_1^k(n), x_b^k(n)(c), x_g^k(n)(c)) - U^\gamma(x_1^k(n), x_g^k(n)(c), x_b^k(n)(c))\big]h_n(c)\,dc \to 0. \tag{20}$$

But as we observed in Part 1, the solution to (15), call it $\overline{\overline{x}}_1^\gamma(n)$, converges to $\overline{x}_1^\gamma$. Given (19) and (20), there is a $\delta > 0$ such that for all sufficiently large n, there is a *feasible* allocation $((x_1^k(n), x_g^k(n), x_b^k(n))_{k\in\mathcal{I}}$, where, for each $k \in \mathcal{I}$,

$$\begin{aligned}\int_{\underline{c}(n)}^{\overline{c}(n)} & U^\gamma(x_1^k(n), x_g^k(n)(c), x_b^k(n)(c))h_n(c)\,dc \\ & \ge \int_{\underline{c}(n)}^{\overline{c}(n)} U^\gamma(\overline{\overline{x}}_1^\gamma(n), \overline{\overline{x}}_b^\gamma(n)(c), \overline{\overline{x}}_g^\gamma(n)(c))h_n(c)\,dc + \delta,\end{aligned}$$

where

$$(\overline{\overline{x}}_b^\gamma(n)(c), \overline{\overline{x}}_g^\gamma(n)(c)) \equiv \big(e_b^\gamma(n) + c(e_1^\gamma(n) - \overline{\overline{x}}_1^\gamma(n)), e_g^\gamma(n)\big)$$

for $c \in [\underline{c}(n), \overline{c}(n)]$. For large n, this contradicts the fact that (16) is solved at $(x_1^1, \dots, x_1^I) = (\overline{\overline{x}}_1^\gamma(n), \dots, \overline{\overline{x}}_1^\gamma(n))$. □

[10] We use $U^\gamma(x_1^i, x_b^i, x_g^i)$ to denote $\min_{\pi\in[\underline{\pi}^\gamma, \overline{\pi}^\gamma]}(\pi u^\gamma(x_1^i, x_b^i) + (1-\pi)u^\gamma(x_1^i, x_g^i))$.

[11] We use $U_n^k(x_1^k, x_b^k, x_g^k)$ to denote $\min_{\pi\in[\underline{\pi}^k(n), \overline{\pi}^k(n)]}(\pi u^k(n)(x_1^k, x_b^k) + (1-\pi)u^k(n)(x_1^k, x_g^k))$.

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