“Topologies on types”: Correction*

Yi-Chun Chen[†] Siyang Xiong[‡]

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Abstract

We show by an example that the Proposition 2 in "Topologies on Types" [3, *Theoretical Economics* **1** (2006) 275–309] is not true.

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[†]Department of Economics, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, johnchen@northwestern.edu

[‡]Department of Economics, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, xiong@u.northwestern.edu

In a recent paper, Dekel, Fudenberg, and Morris [3, hereafter, DFM] propose the strategic topology which is defined to be just strong enough to guarantee that the correspondence mapping types into ε-interim-correlated-rationalizable actions is continuous. That is, two types are close under the strategic topology if and only if they have similar ε-interim-correlated-rationalizable actions in every finite game. They show that the strategic topology is still weak enough that finite types are dense in the universal type space.

In contrast to the strategic topology, DFM also consider the *uniform* strategic topology which requires the degree of similarity of strategic behavior to be uniform over all finite games. DFM argue by their Proposition 2 that finite types are not dense under the uniform strategic topology. In this note, we present a counterexample to show that a step in their proof is not correct.[1] We also fill a gap in their proof of Proposition 2.

In order to make our discussion self-contained, we briefly define the following notation. For any topological space Y, let $\Delta(Y)$ be the space of Borel probability measures on Y endowed with the standard weak* topology. Let $Y^0 = \Theta$ be the finite set of basic uncertainty endowed with the discrete topology. For every $k \geq 1$, let $Y^k = Y^{k-1} \times \Delta\left(Y^{k-1}\right)$. Let (T^*, π^*) be the resulting Mertens-Zamir universal type space where $T^* \subset \times_{k=0}^{\infty} \Delta\left(Y^k\right)$ and π^* is the homeomorphism between T^* (endowed with the product topology) and $\Delta(\Theta \times T^*)$. For $i = 1, 2$, let $T_i^* = T^*$ and $\pi_i^* = \pi^*$. For any $y \in Y$, let δ_y denote the Dirac measure on y.

Let $G = (A_i, g_i)_{i=1,2}$ be a finite game where A_i is a finite set of actions and $g_i : A_1 \times A_2 \times \Theta \to [-1, 1]$ is the payoff function for player i. For any $\varepsilon \geq 0$, DFM define the ε-interim-correlated-rationalizable set $R(G, \varepsilon)$ to be the largest (w.r.t. set inclusion) set in $\left(\left(2^{A_i}\right)^{T_i^*}\right)_{i=1,2}$ with the best reply property that for any $i = 1, 2$, $j = 3 - i$, and $a_i \in R_i(t_i, G, \varepsilon)$, there exists $\nu \in \Delta\left(A_j \times \Theta \times T_j^*\right)$ such that

$$\begin{aligned} \nu\left[\{(a_j, \theta, t_j) : a_j \in R_j(t_j, G, \varepsilon)\}\right] &= 1; \\ \operatorname{marg}_{\Theta \times T_j^*} \nu &= \pi_i^*[t_i]; \\ \int_{(a_j, \theta, t_j)} \left[g_i(a_i, a_j, \theta) - g_i(a_i', a_j, \theta)\right] d\nu &\geq -\varepsilon \text{ for all } a_i' \in A_i. \end{aligned}$$

For each $t_i \in T_i^*$, define $h_i(t_i | a_i, G) = \min\{\varepsilon : a_i \in R_i(t_i, G, \varepsilon)\}$.

[1] In Chen and Xiong [2], we nonetheless confirm their conclusion by explicitly constructing a type which is not the limit of any sequence of finite types under the uniform strategic topology.

The purpose of DFM's Proposition 2 is to establish the equivalence between two metrics d^{US} and d^{**} on T_i^* which are defined as follows. For $t_i, t_i' \in T_i^*$,

$$d^{US}(t_i, t_i') \equiv \sup_{a_i \in A_i(G), G} |h_i(t_i | a_i, G) - h_i(t_i' | a_i, G)|;$$

$$d^{**}(t_i, t_i') \equiv \sup_k \sup_{f \in F_k} |E(f | \pi^*[t_i]) - E(f | \pi^*[t_i'])|,$$

where F_k is the collection of bounded real-valued functions on $\Theta \times T^*$ that are measurable with respect to k^{th}-order beliefs. In particular, they aim to show d^{US} convergence implies d^{**} convergence so that an argument in Morris [5] can be invoked to show that finite types are not dense under d^{US}.

First, we present an example showing that $d^{US}(t^n, t) \to 0$ does not necessarily imply $d^{**}(t^n, t) \to 0$. Let $\Theta = \{0, 1\}$. Consider a hierarchy $t = (\mu_1, \mu_2, \mu_3 ...)$ where it is common 1-belief that $\theta = 0$. Let $t^n = (\mu_1^n, \mu_2^n, \mu_3^n ...)$ be a hierarchy under which both players believe $\theta = 0$ with probability $1 - 1/n$ and it is common 1-belief that both players believe $\theta = 0$ with probability $1 - 1/n$. Hence, $\pi^*[t] = \delta_{(0,t)}$ and $\pi^*[t^n] = (1 - 1/n)\delta_{(0,t^n)} + (1/n)\delta_{(1,t^n)}$ (cf. Mertens and Zamir [4]). Now consider the measurable function $f : \Delta(\Theta) \to [0, 1]$ such that $f(\mu_1) = 1$ if $\mu_1 = \delta_{\{\theta=0\}}$ and $f(\mu_1) = 0$ otherwise. Observe that f can be identified with a bounded function $f^* : \Theta \times T^* \to [0, 1]$ by defining $f^*(\theta, \widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3, ...) = f(\widetilde{\mu}_1)$ for every $(\theta, \widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3, ...)$ in $\Theta \times T^*$. Hence, the value of f^* depends only on $\Delta(\Theta)$ and f^* is measurable with respect to $\Delta(\Theta)$, i.e., $f^* \in F_1$. Observe that $E(f^* | \pi^*[t]) = 1$ and $E(f^* | \pi^*[t^n]) = 0$ for every n. Therefore, $|E(f^* | \pi^*[t]) - E(f^* | \pi^*[t^n])| = 1$ and hence $d^{**}(t^n, t) \geq 1$ for every n. However, it is straightforward to verify that the Prohorov metric between the k^{th}-order beliefs of t^n and t equals $1/n$ for every n and $k \geq 1$, which can be used to show that $d^{US}(t^n, t) \to 0$.[2]

Second, DFM also show that $d^{**}(t_i, t_i') \to 0$ implies $d^{US}(t_i, t_i') \to 0$. They start with two types t_i and t_i' with $d^{**}(t_i, t_i') \leq \varepsilon$ and aim to show that $R_i(t_i, G, \gamma) \subseteq R_i(t_i', G, \gamma + 4\varepsilon)$ for any $\gamma \geq 0$, which implies $d^{US}(t_i, t_i') \leq 4\varepsilon$. However, for $a_i \in R_i(t_i, G, \gamma)$, when DFM choose a conjecture ν' to $(\gamma + 4\varepsilon)-$rationalize a_i for t_i', they do not explicitly check if $\nu'[\{(a_j, \theta, t_j) : a_j \in R_j(t_j, G, \gamma + 4\varepsilon)\}] = 1$ is true. We propose one way to deal with this issue. Suppose that $a_i \in R_i(t_i, G, \gamma)$ and ν is a conjecture which $\gamma-$rationalizes a_i. Since $A_j \times \Theta \times T_j^*$ is a standard separable measure space, there exist conditional probabil-

[2] A detailed proof is provided in Chen and Xiong [2].

ities $\nu(\cdot|\theta, t_j) \in \Delta(A_j)$. Also, since $t_j \mapsto R_j(t_j, G, \gamma + 4\varepsilon)$ is upper hemicontinuous under the product topology on T_j^*, by Kuratowski-Ryll-Nardzewski Theorem (see [1]), there is a measurable function $d : T_j^* \to A_j$ with $d(t_j) \in R_j(t_j, G, \gamma + 4\varepsilon)$ for all $t_j \in T_j^*$. Let $S^* = \{(\theta, t_j) : \text{support}[\nu(\cdot|\theta, t_j)] \subseteq R_j(t_j, G, \gamma)\}$. To define ν', we first define a measurable function $b_j : \Theta \times T_j^* \to \Delta(A_j)$ as

$$b_j(\theta, t_j) = \begin{cases} \nu(\cdot|\theta, t_j), & \text{if } (\theta, t_j) \in S^*; \\ \delta_{d(t_j)} & \text{if } (\theta, t_j) \notin S^*. \end{cases}$$

Then, define the conjecture $\nu' \in \Delta\left(A_j \times \Theta \times T_j^*\right)$ such that for any measurable set $E \subseteq T_j^*$ and $(a_j, \theta) \in A_j \times \Theta$, $\nu'(E \times \{(a_j, \theta)\}) \equiv \int_E b_j(a_j|\theta, t_j)\, \pi_i^*(t_i')\,[(\theta, dt_j)]$. Observe that $\text{marg}_{\Theta \times T_j^*}\nu' = \pi_i^*[t_i']$. Moreover, we have $\nu'[\{(a_j, \theta, t_j) : a_j \in R_j(t_j, G, \gamma + 4\varepsilon)\}] = 1$, because support$[b_j(\theta, t_j)] \subseteq R_j(t_j, G, \gamma + 4\varepsilon)$ for all $t_j \in T_j^*$ by the definitions of S^* and $d(\cdot)$. Then, we can use equation (8) in DFM [3, p.306] to verify that a_i is a $(\gamma + 4\varepsilon)-$best reply to ν'.[3] Therefore, $a_i \in R_i(t_i', G, \gamma + 4\varepsilon)$ and $d^{US}(t_i, t_i') \leq 4\varepsilon$.

References

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[5] S. Morris (2002): "Typical Types," mimeo, Princeton University.

[3] A detailed proof is provided in Chen and Xiong [2].